MODELLING OF CONTACT PROBLEMS

DISSERTATION

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IN

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BY

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In the Name of Allah, the Merciful, the Compassionate

All Praise be to Allah, Lord of all the Worlds

DEDICATED

TO

MY PARENTS

C E R T I F I C A T E

Certified that Mr. Shamshad Husain has carried out the research on 'Modelling of Contact Problems' under my supervision and the work is suitable for submission for the award of the degree of Master of Philosophy in Mathematics.

Asiddiqi

(PROF. A.H. SIDDIQI)
Supervisor

A C K N O W L E D G E M E N T S

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Husain

(SHAMSHAD HUSAIN)

PREFACE

In virtually every structural and mechanical system, there exists a situation in which one deformable body comes in contact with another. In fact, the contact of one body with another is, in essence, how loads are delivered to a structure and is the mechanism whereby structures are supported to sustain loads. It is obvious, therefore, that the character of this contact may play a fundamental role in the behaviour of the structure: its deformation, its motion, the distribution of stresses, etc.

In the recent past serious attempts have been made to resolve these problems through the techniques of variational inequalities. The beauty of this technique lies in the fact that the unknown boundary becomes an intrinsic part of the variational inequality problem.

Contact problems are inherently nonlinear. Prior to the application of loads to a body, the actual contact surface on which bodies meet is unknown. The boundary conditions on this unknown surface involve unknown stresses and displacements. As a result, mathematical models of contact involve systems of inequalities or nonlinear equations. Moreover, when friction is present, multiple solutions of the equations describing contact can exist, and the description of the motion of the bodies in contact becomes extremely complex.

The main object of this dissertation is to discuss the analysis of contact problems in solid mechanics with the emphasis on the problem of equilibrium of elastic bodies in contact with both rigid and deformable foundations.

There are five chapters in this dissertation. Chapter I is devoted to some fundamental results of Applied Functional Analysis which are required for the presentation of results in the subsequent chapters. In Chapter II we discuss Signorini's Problem in elasticity: The equilibrium of a linearly elastic body in contact with a rigid frictionless foundation. A physical meaning, mathematical formulation in terms of variational inequalities and its existence of solutions are discussed along with finite element approximation of this problem. Chapter III is devoted to Signorini's Problem with friction. A study of a rigid punch problem of a linearly elastic incompressible body is presented in Chapter IV while Chapter V deals with a unilateral contact problem in linear elasticity.

At the end a fairly comprehensive and uptodate bibliography has been given.

C O N T E N T S

CHAPTER-I

PRELIMINARIES

1.1 INTRODUCTION:

In this preliminary chapter we briefly mention some results of Applied Functional Analysis which are essential for presentation of the results in the subsequent chapters. For the details of the results in this chapter we refer to [01], [14] and [34]. Some basic results concerning Sobolev spaces are discussed in Section 1.2 while Section 1.3 is devoted to boundary spaces, trace results in elasticity problems and Korn's inequalities. Some important results on minimization of functionals are mentioned in Section 1.4. Variational inequalities in Hilbert space are discussed in Section 1.5. In Section 1.6, we have discussed finite element method.

1.2 SOBOLEV SPACES:

DEFINITION 1.2.1: The **Sobolev space of order 1** on Ω (an open subset of R^N), denoted by $H^1(\Omega)$ is defined by

$$H^1(\Omega) = \left\{ f \in L_2(\Omega) : \frac{\partial f}{\partial x_i} \in L_2(\Omega),\ 1 \leq i \leq N \right\} \qquad (1.2.1)$$

where $\frac{\partial f}{\partial x_i}$ are distributional derivatives.

REMARK 1.2.1: (i) If $n = 1, \Omega = [a,b]$, $f : [a,b] \longrightarrow R$ then

$$H^1(a,b) = \left\{ f \in L_2(a,b) : \frac{\partial f}{\partial x} \in L_2(a,b) \right\} \tag{1.2.2}$$

(ii) $f \in L_2(\Omega)$ need not imply $\frac{\partial f}{\partial x_i} \in L_2(\Omega)$. for example:

$$f(x) = \begin{cases} -1 \ , & x < 0 \\ 0 \ , & x \geq 0 \end{cases} \quad , \ \Omega = [-1,1]$$

$f \in L_2(\Omega)$, but $\frac{df}{dx} = \phi(0) \notin L_2(\Omega)$.

THEOREM 1.2.1 [34]: $H^1(\Omega)$ is a Hilbert space with respect to the inner product,

$$\langle f,g \rangle = \langle f,g \rangle_{1,L_2(\Omega)} + \sum_{i=1}^{N} \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i} \right\rangle_{L_2(\Omega)} \tag{1.2.3}$$

REMARK 1.2.2: If $\Omega = [a,b]$, then $H^1(a,b)$ is a Hilbert space with respect to the inner product

$$\langle f,g \rangle = \int_a^b f(x)g(x)\,dx + \int_a^b \frac{df}{dx}\frac{dg}{dx}\,dx \,. \tag{1.2.4}$$

DEFINITION 1.2.2: For any integer k and $1 \leq p \leq \infty$ the Sobolev space (**generalized**) is the space of all functions

$f \in L_p(\Omega)$ whose all derivatives in the sense of distribution of order $\leq k$ also belong to $L_p(\Omega)$, that is

$$W^{k,p}(\Omega) = \left\{ f \in L_p(\Omega) : D^{\alpha} f \in L_p(\Omega), \text{ for all } \alpha, |\alpha| \leq m \right\} \tag{1.2.5}$$

$W^{k,p}(\Omega)$ is a Banach space with the norm,

$$\| f \|_{W^{k,p}} = \Big[\sum_{|\alpha| \leq m} \| D^{\alpha} f \|^{p}_{L_p(\Omega)} \Big]^{1/p} . \tag{1.2.6}$$

For $p = 2$, we write $W^{k,2}(\Omega) = H^{k}(\Omega)$.

$H^k(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle f,g \rangle_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \langle D^{\alpha} f, D^{\alpha} g \rangle_{L_2(\Omega)}$$

$$= \int_{\Omega} \Big[\sum_{|\alpha| \leq k} D^{\alpha}(f)\ D^{\alpha}(g) \Big] dx \tag{1.2.7}$$

The norm induced by this inner product is

$$\| f \|_{H^k(\Omega)} = \Big[\langle f,f \rangle_{H^k(\Omega)} \Big]^{1/2} \tag{1.2.8}$$

$H^k(\Omega)$ is called a Sobolev space of order k.

$$H_0^k(\Omega) = \left\{ f : f \in H^k(\Omega) \text{ and } \left. \frac{\partial^m f}{\partial n^m} \right|_{\Gamma} = 0,\ 0 \le m \le k-1 \right\} \tag{1.2.9}$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative on the boundary Γ.

THEOREM 1.2.2 [34] (Green's formula): If $v, \phi \in H^1(R_+^N)$,

where $R_+^N = \left\{ x = (x_1, x_2, \ldots\ldots, x_N) : x_N > 0 \right\}$

then

$$\int_{R_+^N} v \frac{\partial \phi}{\partial x_i}\, dx = - \int_{R_+^N} \frac{dv}{dx_i} \phi\, dx, \quad 1 \le i \le N-1 \tag{1.2.10}$$

$$\int_{R_+^N} v \frac{\partial \phi}{\partial x_i}\, dx = - \int_{R_+^N} \frac{dv}{dx_n} \phi\, dx + \int_{R^{N-1}} v(x_1, x_2, \ldots, x_{N-1}, 0) \cdot \phi(x_1, x_2, \ldots, x_{N-1}, 0)\, dx^1 \tag{1.2.11}$$

where $x^1 = (x_1, x_2, \ldots\ldots, x_{N-1})$.

THEOREM 1.2.3 [34] (Generalized Green's formula): Let Ω be a bounded open subset with sufficient regular boundary Γ. Then for $u, v \in H^1(\Omega)$, we have

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v\, dx = - \int_{\Omega} u \frac{\partial v}{\partial x_i}\, dx + \int_{\Gamma} u v n_i\, d\Gamma . \tag{1.2.12}$$

1.3 BOUNDARY SPACES AND TRACE THEOREMS:

The present section deals with several properties of boundary spaces of $H^1(\Omega)$- functions.

Let Ω be a bounded open domain in R^N (N = 1,2 or 3). If Ω is connected with points on only one side of its boundary Γ. Then the **smoothness of** Γ is measured by the following scheme:

(i) We cover Γ with a collection $\{U_1, U_2, \ldots, U_M\}$ of open subsets of R^N, $\Gamma \subset \bigcup_{r=1}^{M} U_r$, such that

$$\Gamma_r = U_r \cap \Gamma \neq \phi , \quad r = 1,2,3,\ldots, M. \tag{1.3.1}$$

(ii) For each U_r, we introduce an orthogonal transformation A_r of the coordinate system $x = (x_1, x_2, \ldots x_N)$ into local coordinate systems $y_r = (y_{r1}, y_{r2}, \ldots, y_{rN})$:

$$y_r = A_r(x), \quad r = 1,2,\ldots,M. \tag{1.3.2}$$

(iii) We assume that there exists an $\alpha > 0$ and a $\beta > 0$ such that, locally, the smoothness of the boundary Γ can be described in terms of hyper-surfaces defined by functions f_r on sets S_r, where

$$S_r = \left\{ y_r' = (y_{r1}, y_{r2}, \ldots\ldots, y_{rN-1}) : |y_{ri}| < \alpha, \right.$$

$$\left. i = 1,2,\ldots\ldots, N-1 \right\},$$

$$\Gamma_r = U_r \cap \Gamma = \left\{ (y_r', f_r(y_r')) : y_r' \in S_r \right\}, \qquad (1.3.3)$$

$$U_r^+ = U_r \cap \Lambda = \left\{ y_r : y_r' \in S_r, f_r(y_r') < y_{rN} < f(y_r') + \beta \right\},$$

$$U_r^- = U_r - \bar{\Lambda} = \left\{ y_r : y_r' \in S_r, f_r(y_r') - \beta < y_{rN} < f_r(y_r') \right\}.$$

(iv) Let $C^{k,\lambda}(\bar{S})$, $0 \leq k \leq \infty$, $0 \leq \lambda \leq 1$, denote the Hölder class of functions defined on the closure of the set $S \subset R^N$. Thus, $C^{k,\lambda}(\bar{S})$ is the subspace of $C^k(\bar{S})$ consisting of those functions ϕ whose derivatives $D^\alpha \phi$ satisfy, for all $0 \leq |\alpha| \leq k$, the Hölder condition with exponent λ, i.e., there is a constant C such that

$$|D^\alpha \phi(x) - D^\alpha \phi(y)| < C \, \| x-y \|^\lambda, \qquad x,y \in S \qquad (1.3.4)$$

where $\| - \|$ denotes the Euclidean norm on R^N. Then the **regularity of** Γ_r is assumed to be determined by choosing k and λ so that

$$f_r \in C^{k,\lambda}(\bar{S}_r), \quad r = 1,2,\ldots\ldots\ldots, M.$$

(v) The class of all such sets satisfying the above conditions is denoted by $\underline{C}^{k,\lambda}$, and if (iv) holds, we say that $\Lambda \in \underline{C}^{k,\lambda}$. If $\Lambda \in \underline{C}^{0,1}$, Λ is said to be a **Lipschitzian domain.**

<u>DEFINITION</u> 1.3.1: Suppose that a construction defined in (i)-(v) above is available which completely characterizes Γ. Then a subset $G \subset \Gamma$ is called a **zero subset** of Γ if the sets $G'_r \subset R^{N-1}$ defined by

$$G'_r = \left\{ y'_r \in S_r : (y'_r, f_r(y'_r)) \in A_r(G) \right\}, \quad r = 1,2,\ldots\ldots, M$$

have zero measure in R^{N-1} for each r. Moreover, we say that a function u is defined almost everywhere on Γ if the points of Γ at which u is not defined form a zero subset of Γ.

<u>DEFINITION</u> 1.3.2 (The spaces $L^p(\Gamma)$, $W^{k,p}(\Gamma)$): Let a function u be defined almost everywhere on Γ. Then u is said to belong $L^p(\Gamma)$, $1 \leq p \leq \infty$, if each of the functions u_r defined by

$$u_r(y'_r) = u(y'_r, f_r(y'_r)), \quad r = 1,2,\ldots\ldots M$$

belongs to $L^p(S_r)$ where S_r is defined in (1.3.3).

The space $L^p(\Gamma)$. $1 \leq p \leq \infty$, is a Banach space when equipped with the norm

$$\|u\|_{0,p,\Gamma} = \left\{ \sum_{r=1}^{M} \int_{S_r} |u_r(y_r')|^p \, dy_r' \right\}^{1/p}. \qquad (1.3.5)$$

In particular, $L^2(\Gamma)$ is a Hilbert space with inner product

$$(u,v)_{0,\Gamma} = \sum_{r=1}^{M} \int_{S_r} u_r(y_r') \, v_r \, (y_r') \, dy_r' . \qquad (1.3.6)$$

Similarly, the space $W^{k,p}(\Gamma)$, $k \geq 0$, $1 \leq p \leq \infty$,$\Gamma \in C^{k-1,1}$, which is a subspace of $L^p(\Gamma)$, is defined as the set of all functions $u \in L^p(\Omega)$ such that $u_r \in W^{k,p}(S_r)$, $r = 1,2,\ldots\ldots,M$. Then $W^{k,p}(\Gamma)$ becomes a reflexive Banach space for $1 < p < \infty$ with the norm

$$\|u\|_{k,p,\Gamma} = \left\{ \sum_{r=1}^{M} \|u_r\|^p_{k,p,S_r} \right\}^{1/p}, \qquad (1.3.7)$$

for any positive real number $k > 0$.

<u>NOTE</u> 1.3.1: Space $L^p(\Sigma)$ or $W^{k,p}(\Sigma)$ are defined as in Definition 1.3.2 for an open subset Σ of Γ . Mostly we have used $L^2(\Gamma)$ and $H^{1/2}(\Gamma)$, that is, $L^p(\Gamma)$ and $W^{k,p}(\Gamma)$ for $p = 2$ and $k = 1/2$.

DEFINITION 1.3.3 [14] (The space $H_{00}^{1/2}(\Sigma)$): To define space $H_{00}^{1/2}(\Sigma)$, we introduce a function ρ on Σ which has the following properties:

(i) ρ is sufficiently smooth,

(ii) ρ is positive on all of Σ,

(iii) ρ vanishes on the boundary $\partial\Sigma$ of Σ,

(iv) ρ vanishes on $\partial\Sigma$ at a rate

$$d = \lim_{x \longrightarrow x_0} \frac{\rho(x)}{d(x,\partial\Sigma)} \neq 0, \quad x_0 \in \partial\Sigma .$$

Then the space $H_{00}^{1/2}(\Sigma)$ is defined as the subspace of $H^{1/2}(\Sigma)$ given by

$$H_{00}^{1/2}(\Sigma) = \left\{ v \in H^{1/2}(\Sigma) : \rho^{-1/2} v \in L^2(\Sigma) \right\} .$$ An inner product and norm on $H_{00}^{1/2}(\Sigma)$ are defined by

$${}_{00}(u,v)_{1/2,\Sigma} = (u,v)_{1/2,\Sigma} + (\rho^{-1/2} u, \rho^{-1/2} v)_{0,\Sigma} ,$$

$${}_{00}\| u \|_{1/2,\Sigma} = \left\{ {}_{00}(u,u)_{1/2,\Sigma} \right\}^{1/2} \tag{1.3.8}$$

$$= \left\{ \| u \|_{1/2,\Sigma}^2 + \| \rho^{-1/2} u \|_{0,\Sigma} \right\}^{1/2} .$$

NOTE 1.3.2: The dual of $H^{1/2}(\Sigma)$ is represented by $H^{-1/2}(\Sigma)$ that is $(H^{1/2}(\Sigma))' = H^{-1/2}(\Sigma)$.

THEOREM 1.3.1 [14] (Trace Theorem I): Let Ω be Lipschitzian domain and let γ be the operator defined by

$$\gamma(v) = v|_{\Gamma} \quad \text{for} \quad v \in C^{\infty}(\bar{\Omega}). \tag{1.3.9}$$

Then γ can be extended to a continuous linear operator, also denoted by γ, from $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$.

REMARK 1.3.1: The operator in Theorem 1.3.1 is called the **trace operator.** A key property of γ is that it is surjective as a map from $H^1(\Omega)$ into $H^{1/2}(\Gamma)$ with $\ker \gamma = H_0^1(\Omega)$.

Let Γ_D denote a non-empty open subset of Γ and let V denote the subspace of $H^1(\Omega)$ given by

$$V = \left\{ v \in H^1(\Omega) : \gamma_D(v) = 0 \right\} \tag{1.3.10}$$

where γ_D is the map which associates $v \in H^1(\Omega)$ with the restriction of $\gamma(v) \in H^{1/2}(\Gamma)$ to Γ_D,

$$\gamma_D : H^1(\Omega) \longrightarrow H^{1/2}(\Gamma_D).$$

Complementary to γ_D, it is also natural to consider the trace operator

$$\gamma_\Sigma^O : V \longrightarrow H^{1/2}(\Sigma). \quad \Sigma = \text{int}\,(\Gamma - \Gamma_D)$$

defined in a manner analogous to γ_D. Properties of γ_Σ^O have been studied by Baiocchi and Capelo [02]. In particular, we note that: The trace operator

$$\gamma_\Sigma^O : V \longrightarrow H_{OO}^{1/2}(\Sigma) \tag{1.3.1}$$

is continuous, linear and surjective for C^∞-boundary $\partial\Sigma$.

TRACE RESULTS FOR ELASTICITY PROBLEMS:

THEOREM 1.3.2 [14]: Let Ω be a Lipschitzian domain. Then a unit vector n, outward and normal to the boundary Γ, exists almost everywhere on Γ, that is, $n_i \in L^\infty(\Gamma)$, $i = 1,2,\ldots\ldots, N$.

It is clear that if Ω be Lipschitzian domain, the normal component of displacement $v_n = v.n$ can be defined on the boundary for $v \in V$. Indeed, if $v_i \in H^1(\Omega)$, then we can set $v_n = \gamma(v_i)n_i$ almost everywhere on Γ. Moreover for every $v \in L^2(\Gamma)$ the decomposition

$$v = v_T + v_n \cdot n, \; v_n = v \cdot n, \; v_T = v - v_n \cdot n \tag{1.3.12}$$

makes sense in $L^2(\Gamma)$, that is,

$$v_n \in L^2(\Gamma) \quad \text{and} \quad v_T \in L_T^2(\Gamma) \tag{1.3.13}$$

where

$$L_T^2(\Gamma) = \left\{ v \in L^2(\Lambda) = (L^2(\Lambda))^N : v_n = 0 \right\}. \tag{1.3.14}$$

Thus, the decomposition $v \longrightarrow \{v_n, v_T\}$ is an isomorphism from $L^2(\Gamma)$ onto $L^2(\Gamma) \times L_T^2(\Gamma)$, and

$$(u,v)_{L^2(\Gamma)} = (u_n, v_n)_{n,\Gamma} + (u_T, v_T)_{T,\Gamma} \tag{1.3.15}$$

where

$$(u,v)_{L^2(\Gamma)} = \int_\Gamma u_i v_i \, ds,$$

$$(u_n, v_n)_{n,\Gamma} = \int_\Gamma u_n v_n \, ds, \tag{1.3.16}$$

$$(u_T, v_T)_{T,\Gamma} = \int_\Gamma u_{T_i} \, v_{T_i} \, ds.$$

Now we consider the decomposition of $H^{1/2}(\Gamma)$ following the plan of $L^2(\Gamma)$. Let $\Lambda \in \underline{C}^{1,1}$, that is, each component n_i

of the unit vector outward normal to the boundary is Lipschitz continuous with uniform constant on Γ. Then, applying properties of products of functions in Sobolev spaces, we have

$$vn_i \in H^{1/2}(\Gamma) \quad \text{if} \quad v \in H^{1/2}(\Gamma). \tag{1.3.17}$$

Thus, for $v \in H^{1/2}(\Gamma) = (H^{1/2}(\Gamma))^n$,

$$v_n = v_i n_i \in H^{1/2}(\Gamma), \tag{1.3.18}$$

$$v_T = v - v_n \cdot n \in H_T^{1/2}(\Gamma),$$

where

$$H_T^{1/2}(\Gamma) = \left\{ v \in H^{1/2}(\Gamma) : v_n = 0 \right\}. \tag{1.3.19}$$

That is, the decomposition $v \longrightarrow \{v_n, v_T\}$ is an isomorphism from $H^{1/2}(\Gamma)$ onto $H^{1/2}(\Gamma) \times H_T^{1/2}(\Gamma)$. Hence

$$v = v_n \cdot n + v_T \; , \; v \in H^{1/2}(\Gamma). \tag{1.3.20}$$

Therefore, the trace theorem, Theorem 1.3.1, can be replaced by the following decomposed trace **theorem:**

<u>THEOREM</u> 1.3.3 [14] (Trace theorem II): Let the domain $\Omega \in \underline{C}^{1,1}$

Then there exist uniquely determined linear continuous maps γ_n of $H^1(\Omega)$ into $H^{1/2}(\Gamma)$ and γ_T of $H^1(\Omega)$ into $H_T^{1/2}(\Gamma)$ such that

$$\gamma(v) = \gamma_n(v).n + \gamma_T(v) \qquad v \in H^1(\Omega) \tag{1.3.21}$$

where

$$\gamma_n(v) = v_n \;, (\gamma_T(v))_i = v_{T_i}, \; v \in C^\infty(\bar{\Omega}).$$

Furthermore, for a given $(h,g) \in H^{1/2}(\Gamma) \times H_T^{1/2}(\Gamma)$, there exist a $v \in H^1(\Omega)$ and a constant C such that

$$\begin{aligned} &\gamma_n(v) = h \quad \text{and} \quad \gamma_T(v) = g, \\ &\| v \|_1 \leq C \, (\| h \|_{1/2,\Gamma} + \| g \|_{1/2,\Gamma}). \end{aligned} \tag{1.3.22}$$

The maps γ_n and γ_T are thus surjective.

In view of Theorem 1.3.3 and considering (1.3.10) and (1.3.11), a similar result to Theorem 1.3.3 can be given as follows:

<u>THEOREM</u> 1.3.4 [14] (Trace theorem III): Let the domain $\Omega \in \underline{C}^{1,1}$. Then there exist surjective linear continuous maps

$$\gamma^{0}_{\Sigma n} : V \longrightarrow H^{1/2}_{00}(\Sigma), \ \gamma^{0}_{\Sigma T} : V \longrightarrow H^{1/2}_{T00}(\Sigma)$$

$$\text{with } \ H^{1/2}_{T00}(\Sigma) = \left\{ v \in H^{1/2}_{00}(\Sigma) : v_n = 0 \right\} \qquad (1.3.23)$$

such that

$$\gamma^{0}_{\Sigma}(v) = \gamma^{0}_{\Sigma n}(v) + \gamma^{0}_{\Sigma T}(v) \qquad v \in V.$$

<u>DEFINITION</u> 1.3.4: Let $v \in H^{1/2}_{00}(\Gamma)$ and let $\overline{\Gamma}_c$ be contained in Σ. Then the function $v \geq 0$ on Γ_c in $H^{1/2}_{00}(\Sigma)$, if there exist a sequence $\{v_m\}$ of Lipschitz continuous functions such that

$$v_m(x) \geq 0, \quad x \in \Gamma_c, \ v_m \longrightarrow v \ \text{ weakly in } \ H^{1/2}_{00}(\Sigma).$$

<u>THEOREM</u> 1.3.5 [14]: Let the domain $\Omega \in \underline{C}^{1,1}$ and let g be given in $H^{1/2}_{00}(\Sigma)$. Then, it is meaningful to define the set

$$K = \left\{ v \in V : \gamma^{0}_{\Sigma n}(v) - g \leq 0 \text{ on } \Gamma_c \text{ in } H^{1/2}_{00}(\Sigma) \right\}. \qquad (1.3.24)$$

Moreover, K is a closed convex subset of $V \subset H^{1}(\Omega)$.

KORN'S INEQUALITIES [39]:

THEOREM 1.3.6 [14]: Let Ω be a possibly unbounded domain in R^N. Then there exists a positive constant C, independent of v, such that

$$\int_\Omega |v_{i,j}\, v_{i,j}|^{p/2}\, dx \leq C \int_\Omega |\epsilon_{ij}(v)\; \epsilon_{ij}(v)|^{p/2}\, dx \qquad (1.3.25)$$

for $1 < p < \infty$ and for every $v \in W_0^{1,p}(\Omega) = (W_0^{1,p}(\Omega))^N$,

where $\epsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i})$, $v_{i,j} = \frac{\partial v_i}{\partial x_j}$, $1 \leq i,j \leq N$.

THEOREM 1.3.7 [14]: Let the domain Ω be bounded and Lipschitzian in R^N. Then there exists a positive constant C, independent of v, such that

$$\int_\Omega |v_{i,j}\, v_{i,j}|^{p/2} dx \leq C \left\{ \int_\Omega |\epsilon_{ij}(v)\quad \epsilon_{ij}(v)|^{p/2} dx + \int_\Omega |v_i v_i|^{p/2} dx \right\} \qquad (1.3.26)$$

for every $v \in W^{1,p}(\Omega)$, $1 < p < \infty$.

1.4 MINIMIZATION OF FUNCTIONALS:

The present section deals with certain standard results on the minimization of functionals on reflexive Banach spaces and some definitions which enable us to specify concrete properties of functionals.

Let V be a reflexive (real) Banach space with norm $\| - \|$, K be a nonempty closed convex subset of V and $F : K \longrightarrow R$ be a real functional defined on K. Then the problem of finding $u \in K$ such that

$$F(u) = \inf F(v), \quad \text{for all } v \in V$$

or

$$F(u) \leq f(v), \quad \text{for all } v \in V \tag{1.4.1}$$

is called the **minimization problem** for the functional F relative to the set K.

<u>DEFINITION</u> 1.4.1: A function $F : K \longrightarrow R$ is said to be **weakly lower semicontinuous** if, for any sequence $\{u_k\}$ in K with the property that $\{u_k\}$ converges weakly to $u \in K$, we have

$$\lim_{k \longrightarrow \infty} \inf F(u_k) \geq F(u). \tag{1.4.2}$$

The functional F is **weakly upper semicontinuous** if

$$\lim_{k \longrightarrow \infty} \sup F(u_k) \leq F(u).$$

If $u_k \longrightarrow u$ strongly in K and (1.4.2) holds, F is said to be **lower semicontinuous.** Similarly we can define upper semicontinuity.

<u>DEFINITION</u> 1.4.2: A functional $F : K \longrightarrow R$ is said to be convex if and only if

$$F(\theta u + (1-\theta)v) \leq \theta F(u) + (1-\theta)F(v),$$

for all $u,v \in K$, $0 \leq \theta \leq 1$.

F is **strictly convex** if the strict inequality ($<$) holds for $u \neq v$. If $-F$ is convex, F is said to be **concave.**

<u>DEFINITION</u> 1.4.3: A functional $F : K \longrightarrow R$ is **Gâteaux differentiable** at a point $u \in K$ if there exists a linear functional $DF(u) \in V'$ (dual of V) such that, for every $v \in K$,

$$\lim_{\epsilon \longrightarrow 0} \frac{\partial}{\partial \epsilon} F(u + \epsilon v) = \langle DF(u), v \rangle$$

where ϵ is an arbitrary positive number and $\langle \cdot, \cdot \rangle$ denotes duality pairing on $V' \times V$. We call $DF(u)$ the gradient of F at u and $\langle DF(u), v \rangle$ the Gâteaux derivative of F at u in the direction v.

<u>DEFINITION</u> 1.4.4: A functional $F : K \longrightarrow R$ is said to be coercive if and only if

$$\lim_{\|v\| \to \infty} F(v) = +\infty \ .$$

<u>THEOREM</u> 1.4.1 [14]: Let F be a real functional defined on a nonempty closed convex subset K of a reflexive Banach space V. Moreover, let either of the following conditions hold:

(i) F is weakly lower semicontinuous on K and K is bounded, or

(ii) F is weakly lower semicontinuous and coercive on K.

Then F attains its minimum vlaue on K, that is, there exists at least one $u \in K$ such that (1.4.1) holds.

<u>THEOREM</u> 1.4.2 [14]: Let F be a real, coercive, convex and Gâteaux differentiable functional defined on a possibly unbounded nonempty, closed and convex subset K of a reflexive Banach spac V. Then F attains its minimum value on K. If F is strictly convex, the minimizer is unique.

<u>DEFINITION</u> 1.4.5: Suppose $F : V \longrightarrow \overline{R}$ is convex but not Gâteaux differentiable functional on V. Then, the set $\partial F(u)$ in V', of all linear functionals f such that

$$F(v) - F(u) \geq \langle f, v-u \rangle \qquad \text{for all} \quad v \in V$$

and $|F(u)| < \infty$, is called the **subdifferential** of F at u and any $f \in \partial F(u)$ is a **subgradient** of F at u.

<u>**THEOREM**</u> 1.4.3 [14]: Let F be a convex functional from a reflexive Banach space V into $\overline{R}$, and let F be finite and continuous at a point $u \in V$. Then F is subdifferentiable on the interior of its effective domain $\text{dom } F = \{v \in V : f(v) < +\infty\}$. In particular, F is subdifferentiable at the point u.

<u>**THEOREM**</u> 1.4.4 [14]: Let K be a subset of a normed linear space V and let F be a Gâteaux differentiable functional mapping K into R. If u is a minimizer of F in K, then u may be characterized in one of the following ways:

(i) If K is nonempty closed convex subset of V, then

$$\langle DF(u), v-u \rangle \geq 0 \qquad \text{for all} \quad u \in K. \qquad (1.4.3)$$

(ii) If K is a nonempty closed convex subset of V and u belongs to interior of K, then

$$\langle DF(u), v\rangle = 0 \quad \text{for all} \quad v \in K. \qquad (1.4.4)$$

NOTE 1.4.1: Inequality (1.4.3) referred to as a variational inequality.

DEFINITION 1.4.6: Let V be a Hilbert space and $a(.,.) : V \times V \longrightarrow R$ a bilinear functional.

(i) Then $a(.,.)$ is called symmetric if

$$a(u,v) = a(v,u) \quad \text{for all} \quad u,v \in V. \qquad (1.4.5)$$

(ii) Then $a(.,.)$ is continuous if there exists a constant $M > 0$ such that

$$a(u,v) \leq M \, \|u\| \, \|v\| \qquad (1.4.6)$$

(iii) Then $a(.,.)$ is coercive if there exists a constant $m > 0$ such that

$$a(u,u) \geq m \, \|\Gamma u\|^2 \quad \text{for all} \quad u \in V. \qquad (1.4.7)$$

DEFINITION 1.4.7 [14]: Let V be a Hilbert space and the functional defined in (1.4.1) be of the form

$$F(v) = \frac{1}{2} a(v,v) - f(v), \qquad v \in V \tag{1.4.8}$$

where $a(.,.)$ is a symmetric, continuous and coercive bilinear form from $V \times V \longrightarrow R$ and f is a bounded linear operator on V. Let Q be another Hilbert space and $B : V \xrightarrow{\text{onto}} Q$ a continuous linear operator satisfying:

$$|Bv| \leq C\|v\| \qquad \text{for all} \quad v \in V \tag{1.4.9}$$

where $|\cdot|$ denotes norm on Q and C is a constant. For g and operator B in Q, we define a set $K \subset V$ by

$$K = \{v \in V : Bv - g \leq 0 \quad \text{in} \quad Q\}. \tag{1.4.10}$$

This K is closed convex subset in V. Then, finding $u \in K$ such that

$$F(u) \leq F(v) \qquad \text{for all} \quad v \in K, \tag{1.4.11}$$

is called the **constrained minimization problem.**

<u>THEOREM</u> 1.4.5 [14]: Let (1.4.8), (1.4.5), (1.4.6), (1.4.7) and (1.4.9) hold and let K of (1.4.10) be nonempty. Then

there exists a unique solution u to the minimization problem (1.4.10). Moreover, this minimizer is the unique solution of the variational inequality

$$u \in K: \qquad a(u, v-u) \geq f(v-u) \qquad \forall\, v \in K.$$

1.5 VARIATIONAL INEQUALITIES:

The mathematical subject known as variational inequalitie was introduced by the Italian Mathematician G. Stampacchia in the early sixties during his investigations of the problems of Mechanics and potential theory specially related to the problems in elasticity with unilateral constraints. In the last two decades this subject has enjoyed a vigorous growth and has attracted the attention of a large number of Mathematicians, Engineers and Physicists. By now there are several standard research monographs and textbooks dealing with various aspects of this theory (see e.g. Duvant and Lions [08], Glowinski, Lions and Trémolieres [11], Kinderlehrer and Stampacchia [16] and Tartar [38] etc.). In this section we mention some basic notions and properties concerning variational inequalities.

Let V be a Hilbert space, $\|\cdot\|$ be the norm induced by the inner product, $a(.,.): V \times V \longrightarrow R$ be bounded bilinear form, K be a nonempty closed convex subset of V and f be a bounded linear functional on V.

(i) The problem of finding $u \in K$ such that

$$F(u) = \inf F(v) \quad \text{for all} \quad v \in V \qquad (1.5.1)$$

where

$$F(v) = \frac{1}{2}\, a(v,v) - f(v)$$

is known as **an abstract minimization problem.**

(ii) To find $u \in K$ such that

$$a(u, v-u) \geq f(v-u), \text{ for all } v \in K, \qquad (1.5.2)$$

is called **a variational inequality problem.** Inequality (1.5.1) is known as variational inequality and u is its solution.

<u>THEOREM</u> 1.5.1 [34]: If $a(.,.)$ is symmetric and coercive then $u \in K$ is a solution of the abstract minimization problem (1.5.1) if and only if u is a solution of the variational inequality (1.5.2).

<u>THEOREM</u> 1.5.2 [34] (Lax-Milgram Lemma): Let V be a Hilbert space and $a(.,.): V \times V \longrightarrow R$ be a bounded bilinear form which is coercive. Also, let $f: V \longrightarrow R$ be bounded linear functional. Then there exists a unique element $u \in V$ such that

$$a(u,v) = f(v), \quad \text{for all} \quad v \in V. \tag{1.5.3}$$

<u>PROBLEM</u> 1.5.1: The problem of finding an element u such that

$$a(u,v) = f(v), \quad \text{for all} \quad v \in V \tag{1.5.4}$$

where $a(.,.)$ and f are same as in Theorem 1.5.2, is known as the **abstract variational problem.**

<u>THEOREM</u> 1.5.3 [34]: (Lions-Stampacchia): If $a(.,.)$ is coercive then the variational inequality (1.5.1) has a unique solution.

<u>REMARK</u> 1.5.1: If we choose $K = V$ in Theorem 1.5.3. Then we get

$$a(u,v) = f(v)$$

which has a unique solution.

1.6 THE FINITE ELEMENT METHOD:

The finite element method is concerned with the construction of finite dimensional subspaces of functional spaces specially Sobolev spaces and study of abstract variational problems and in general, Variational Inequalities on such subspaces. This reduces BVPs to the study of matrix equations and inequalities which can be handled through computer. Structural engineers like Argyris, Turner, Clough, Martin and Top have been using the finite element method in structural analysis since 1954. However a mathematical breakthrough came in the paper of Zlamal in 1968 and a sound mathematical theory of this topic has been developed in the last two decades.

Let V_h be a finite dimensional subspace of a given Hilbert space V. Then finding $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h), \quad \text{for all} \quad v_h \in V_h \qquad (1.6.1)$$

is known as **approximate variational problem.** Both the problems (1.5.4) and (1.6.1) possess unique solutions if $a(\cdot,\cdot)$ is bilinear, continuous and coercive [34].

THEOREM 1.6.1 [34] (Céas Lemma): If u and u_h denote the solutions of (1.5.4) and (1.6.1) respectively, there exists a constant C independent of the subspace V_h such that

$$\| u-u_h \|_V \leq C \inf \| u-v_h \|_V \quad \text{for all} \quad v_h \in V_h \qquad (1.6.2)$$

REMARK 1.6.1: The inequality (1.6.2) shows that the problem of estimating the error $\|u-u_h\|$ is reduced to a problem in approximation theory, namely, the evaluation of the distance $d(u,V_h) = \inf \|u-v_h\|$ between a function $u \in V$ and the subspace V_h of V.

COROLLARY 1.6.1: Suppose there exists a dense subspace U of V and a mapping $\gamma_h : U \longrightarrow V_h$ such that $\lim_{h \to 0} \| v-\gamma_h v \| = 0$ for all $v \in U$. Then

$$\lim_{h \to 0} \| u-u_h \| = 0 \qquad (1.6.3)$$

DEFINITION 1.6.1: Let $\Omega \subset R^2$ be a polygonal domain. Then a finite collection of triangles T_h is called a triangulation if the following conditions are satisfied:

(i) $\bar{\Omega} = \bigcup_{K \in T_h} \bar{K}$, $\bar{K}$ denotes a triangle with boundary.

(ii) $K \cap K_1 = \phi$ for $K, K_1 \in T_h$, $K \neq K_1$.

(iii) $\bar{K} \cap \bar{K}_1$ = a vertex or a side, that is, if we consider two triangles, their boundaries may have one vertex common or one side common.

<u>REMARK</u> 1.6.2: Let $P(K)$ be a function space defined on $K \in T_h$ such that $P(K) \subset H^1(K)$. Generally, $P(K)$ will be space of polynomials of some degree.

<u>THEOREM</u> 1.6.2 [34]: Let $C^o(\Omega)$ be the space of continuous real-valued functions on $\bar{\Omega}$ and

$$V_h = \left\{ v_h \in C^o(\bar{\Omega}) \ : \ v_h|K \in P(K), \quad K \in T_h \right\}, \qquad (1.6.4)$$

where $v_h|K$ denotes the restriction of v_h on K and $P(K) \subset H^1(K)$; then $V_h \subset H^1(\Omega)$.

<u>REMARK</u> 1.6.3: Let $h = \max\limits_{K \in T_h}$ (diameter of K),

$N(h)$ = the number of nodes of the triangulation,

$P(K) = P_1(K)$ = space of polynomials of degree less than or equal to 1 in x and y,

$$V_h = \left\{ v_h \ : \ v_h|K \in P_1(K), \ K \in T_h \right\}. \qquad (1.6.5)$$

(i) It can be seen that $V_h \subset C^o(\bar{\Omega})$.

(ii) The functions w_i, $i = 1,2,\ldots\ldots, N(h)$, defined by

$$w_i = \begin{cases} 1 & \text{at the ith node} \\ 0 & \text{at other nodes} \end{cases}$$

form a basis of V_h.

(iii) In view of (ii) and Theorem 1.6.2, V_h defined in this remark is a subspace of $H^1(\Omega)$ of dimension $N(h)$.

ERROR ESTIMATE FOR VARIATIONAL INEQUALITIES:

Let V_h be a finite dimensional subspace of the space V and let K_h be a nonempty closed convex subset of V_h. In general, the set K_h is not a subset of K. Then the Approximate Variational Inequality consists in finding an element $u_h \in K_h$ such that

$$a(u_h, v_h - u_h) \geq f(v_h - u_h), \quad \text{for all} \quad v_h \in K_h. \tag{1.6.6}$$

This variational inequality has a unique solution u_h.

Let A be a bounded linear operator on V into V' defined by the relations

$$\langle Av, w\rangle = a(v,w), \quad \text{for all} \quad v, w \in V \tag{1.6.7}$$

Let H be a Hilbert space with norm $\|-\|_1$ and inner product $\langle .,.\rangle_1$ such that

$$\overline{V} = H \text{ and } V \hookrightarrow H. \tag{1.6.8}$$

The space H will be identified with its dual so that it may be intern identified with a subspace of the dual space of V.

<u>THEOREM</u> 1.6.3 [34]: Let u and u_h denote solutions of (1.5.2) and (1.6.6) respectively and $Au - f \in H$. Then there exists a constant C independent of the subspace V_h and of the set K_h such that

$$\|u-u_h\| \leq C\left(\inf_{v_h \in K_h} \|u-v_h\|^2 + \|Au-f\|_1 \|u-v_h\|_1 + \|Au-f\|_1 \inf_{v \in K} \|u_h - v\|_1\right)^{1/2}$$
(1.6.9)

<u>REMARK</u> 1.6.4: If $K = V$ then $Au - f = 0$ and consequently (1.6.9) reduces to (1.6.2).

<u>THEOREM</u> 1.6.4 [34] (Convergence Theorem): Let $K_h \subset K$ and there exists $U \subset V$ dense in K ($\overline{U} = K$) and $\gamma_h : U \longrightarrow K_h$ such that

$$\lim_{h \to 0} \| \gamma_h v - v \|_V = 0 \quad \text{for all} \quad v \in U. \tag{1.6.10}$$

Then $$\lim_{h \to 0} \| u - u_h \|_V = 0 \tag{1.6.11}$$

where u and u_h are solutions of (1.5.2) and (1.6.6) respectively.

CHAPTER-II

SIGNORINI'S PROBLEM IN LINEAR ELASTICITY

2.1 INTRODUCTION:

A classical example of the role of variational inequalities in the formulation of contact problems in solid mechanics and the basis for many generalizations which we discuss in subsequent chapters, is the so called Signorini problem describing the contact of a linearly elastic body with the rigid frictionless foundation.

In the present chapter we describe the equations and inequalities governing the equilibrium of an elastic body undergoing small deformations whose motion is constrained by the presence of a rigid, lubricated, frictionless foundation which may come in contact with the body during its motion. We construct the contact conditions and associated inequalities, and we then comment on generalizations.

We shall pay special attention to the variational aspects of the problem and we will show that the solution of the classical Signorini problem is also the solution of a variational inequality that arises naturally from the principle of virtual work. Briefly finite element approximation of the variational problem is also discussed. Results discussed in this chapter are mainly due to Kikuchi and Oden [14].

2.2 CONTACT CONDITIONS:

In the present section we discuss the kinematical contact condition for finite displacements of a body constrained by a fixed rigid foundation and then arrive at the general contact conditions for the case of a frictionless rigid foundation. Also we remark linearized contact conditions.

We consider a material body to be the closure of a set Λ of material particles. We describe the motion of the body relative to a fixed spatial frame of reference. The motion of the body in this frame is then characterized by equations giving the position y_i of each particle at each time $t \geq 0$:

$$y_i = \chi_i(x_1, x_2, x_3, t), \quad 1 \leq i \leq 3, \quad (x_1, x_2, x_3) \in \Lambda \tag{2.2.1}$$

cartesian components of displacement of $x = (x_1, x_2, x_3)$ relative to the fixed spatial frame at time t are given by the equations

$$u_i(x_1, x_2, x_3, t) = \chi_i(x_1, x_2, x_3, t) - x_i,$$
$$1 \leq i \leq 3, \quad t \geq 0, \ (x_1, x_2, x_3) \in \Lambda. \tag{2.2.2}$$

Assume that the functions χ_i are differentiable relative to x_i and t, then

$$\det\left(\frac{\partial \chi_i}{\partial x_j}\right) > 0 \quad \text{for all } (x_1,x_2,x_3)\in\bar{\Omega},\ t \geq 0$$

and (2.2.1) can be inverted to give

$$x_i = \chi_i^{-1}(y_1,y_2,y_3,t), \quad 1 \leq i \leq 3. \qquad (2.2.3)$$

Now we consider the geometrical situation Figure 2.2.1, where the body is located in such a manner that $y_3 \geq x_3$.

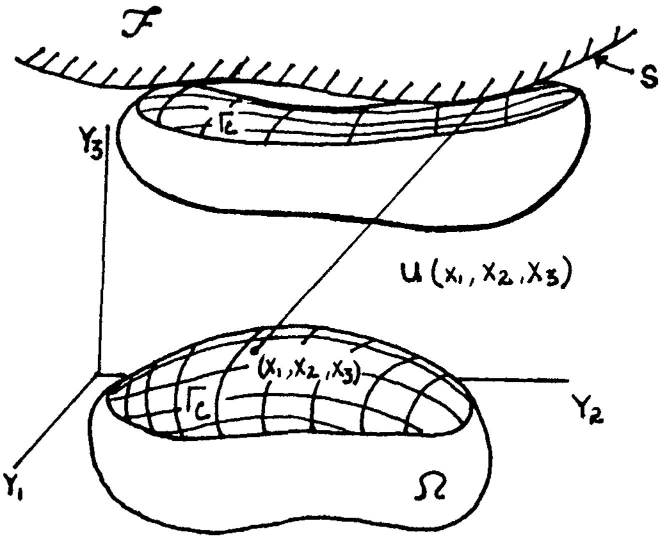

FIG. 2.2.1: Constraint on the motion of a deformable body by a rigid foundation $\mathcal{F}$.

To simplify matters,suppose surface S of the foundation is defined by $y_3 = \psi(y_1,y_2)$ and contact surface Γ_c is defined by $x_3 = \phi(x_1,x_2)$. Assume that ψ and ϕ are smooth functions and $\psi(y_1,y_2) \geq \phi(y_1,y_2)$ for all points

(y_1, y_2) in the projection of Γ_c on the y_1, y_2-plane. Then, if (x_1, x_2, x_3) is a particle on Γ_c, its displacement must satisfy

$$u_i(x_1, x_2, \phi(x_1, x_2)) = \chi_i(x_1, x_2, \phi(x_1, x_2)) - x_i, \quad i = 1,2 \tag{2.2.4}$$

and

$$\phi(x_1, x_2) + u_3(x_1, x_2, \phi(x_1, x_2))$$
$$\leq \psi(x_1, u_1(x_1, x_2, \phi(x_1, x_2)), x_2 + u_2(x_1, x_2, \phi(x_1, x_2))). \tag{2.2.5}$$

The inequality (2.2.5) is called kinematical contact condition. Rewriting (2.2.4) in the form

$$y_i = x_i + u_i \ (x_1, x_2, \phi(x_1, x_2)), \ i = 1,2,$$

and supposing that these conditions can be inverted to give

$$x_i = f_i(y_1, y_2), \quad i = 1,2,$$

(2.2.5) can be written as

$$\hat{\phi}(y_1, y_2) + \hat{u}_3(y_1, y_2) \leq \psi(y_1, y_2) \tag{2.2.6}$$

where

$$\hat{\phi}(y_1, y_2) = \phi(f_1(y_1, y_2), f_2(y_1, y_2)),$$
$$\hat{u}_3(y_1, y_2) = u_3(f_1(y_1, y_2), f_2(y_1, y_2), \phi(f_1(y_1, y_2), f_2(y_1, y_2))),$$

$$\Psi(y_1,y_2) = \Psi(x_1+u_1,x_2+u_2). \qquad (2.2.7)$$

Condition (2.2.6) must also be compatible with the state of stress on the contact surface Γ_c.

Let $T = T(y_1,y_2,y_3)$ denote the Cauchy stress tensor at the particle (x_1,x_2,x_3) whose position is (y_1,y_2,y_3) and let $n = (n_1,n_2,n_3)$ denote outward normal to Γ_c. If $T_{ij}(y)$ are the cartesian components of T of a particle on the material surface Γ are respectively,

$$\begin{aligned} T_n(y) &= T_{ij}(y)n_i(y)n_j(y) \\ T_{T_i}(y) &= T_{ij}(y)n_j(y)-T_n(y)n_i \qquad (2.2.8) \\ x &= \chi^{-1}(y), \quad x \in \Gamma_c, \ 1 \leq i, \ j \leq 3 \end{aligned}$$

where $\chi = (\chi_1, \chi_2, \chi_3)$ and usual summation convention on repeated indices are used. Next we make two fundamental observations:

(i) If no tractions are applied on Γ_c, a compressive normal stress must be developed at the points of contact, T_n is zero if no contact occurs,

(ii) if the foundation surface S is frictionless, the tangential components of stress T_{T_i} at $x \in \Gamma_c$ must be zero. Thus, for $x \in \Gamma_c$ we must have,

$$T_n(y) = 0 \quad \text{if} \quad \hat{\phi}(y_1,y_2)+\hat{u}_3(y_1,y_2) < \psi(y_1,y_2),$$

$$T_n(y) \leq 0 \quad \text{if} \quad \hat{\phi}(y_1,y_2)+\hat{u}_3(y_1,y_2) = \psi(y_1,y_2),$$

$$T_{T_i}(y) = 0 \quad 1 \leq i \leq 3, \; x = \chi^{-1}(y) \in \Gamma_c. \tag{2.2.9}$$

As (2.2.6) and (2.2.9) hold in every case, we must have

$$T_n(y) \; (\hat{\phi}(y_1,y_2)+\hat{u}_3(y_1,y_2) - \psi(y_1,y_2)) = 0 \tag{2.2.10}$$

for all $x = \chi^{-1}(y)$ in Γ_c.

Collecting the results (2.2.6), (2.2.9) and (2.2.10), we arrive at the **general contact conditions:**

$$\begin{aligned} &\hat{\phi}(y_1,y_2)+\hat{u}_3(y_1,y_2) \leq \psi(y_1,y_2) \\ &T_n(y) \leq 0, \quad T_{T_i}(y) = 0 \\ &T_n(y) \; (\hat{\phi}(y_1,y_2)+\hat{u}_3(y_1,y_2)- \psi(y_1,y_2)) = 0 \end{aligned} \tag{2.2.11}$$

for all $x = \chi^{-1}(y) \in \Gamma_c$.

<u>REMARK</u> 2.2.1: Contact conditions in (2.2.11) can also be written in terms of the particles x for infinitesimal motions,

$$u_n(x) - g(x) \le 0,$$

$$T_{T_i}(x) = 0, \quad T_n(x) \le 0 \;, \tag{2.2.12}$$

$$T_n(x)\;(u_n(x) - g(x)) = 0,$$

for $x \in \Gamma_c$, where

$$T_T = T.n - T_n.n, \quad u_n(x) = u(x).n(x) = u_i(x)n_i(x)$$

$$T_n(x) = T_{ij}(x)\; n_i n_j, \quad 1 \le i,j \le 3.$$

Conditions in (2.2.12) are called linearized contact conditions. For the details we refer to [14].

2.3 SIGNORINI'S PROBLEM:

We begin by considering the deformations of an elastic body unilaterally supported by a rigid frictionless foundation, as shown in the Figure 2.3.1, and subjected to body forces $f = (f_1, f_2, f_3)$ and surface tractions $t = (t_1, t_2, t_3)$ applied to a portion Γ_F of the body's surface Γ. Body is fixed along a portion Γ_D of its boundary and we denote by Γ_c a candidate contact surface.

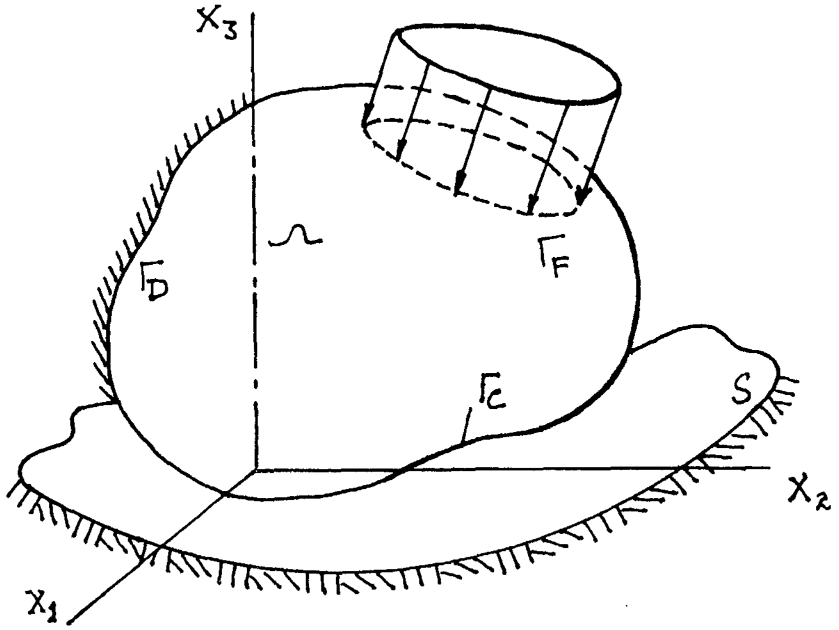

FIG. 2.3.1: An elastic body in contact with a rigid frictionless foundation.

The actual surface on which the body comes in contact with the foundation is not known in advance but is contained in the portion Γ_c of Γ . We confine our attentions to infinitesimal deformations of the body.

Let $V = (v_i) = (v_1,v_2,v_3)$ and $T = (T_{ij})$, $1 \leq i,j \leq 3$, denote arbitrary displacement and stress fields in the body. A stress field $T = T(x)$ is in equilibrium at a particle x on the interior of Ω if

$$- T_{ij}(x)_{,j} = f_i(x), \quad x \in \Omega \tag{2.3.1}$$

where $T_{ij,j} = \frac{\partial}{\partial x_j} (T_{ij})$, $1 \leq i,j \leq 3$.

A displacement field $v = v(x)$ satisfies the kinematical boundary conditions on Γ_D if

$$v_i(x) = 0, \quad x \in \Gamma_D . \tag{2.3.2}$$

Similarly, for traction t applied on Γ_F, the stress produced here must satisfy

$$T_{ij} n_j = t_i \quad \text{on } \Gamma_F. \tag{2.3.3}$$

The components $\epsilon_{ij}(v)$ of the infinitesimal strain tensor ϵ produced by a displacement field v are given by

$$\epsilon_{ij}(v)(x) = \frac{1}{2}\,(v_{i,j}(x)+v_{j,i}(x)), \quad x \in \Omega \tag{2.3.4}$$

where $v_{i,j} = \frac{\partial}{\partial x_j} v_i$.

The mechanical properties of the body are characterized by a constitutive equation given as

$$T_{ij}(v) = \sigma_{ij}(x, \nabla v) = \hat{\sigma}_{ij}(x, \tfrac{1}{2}(v_{i,j}(x)+v_{j,i}(x))) \tag{2.3.5}$$

where T_{ij} depends upon x and ∇v denote displacement gradient tensor at x.

Now, let u denote a specific displacement of the body that corresponds to an equilibrium state of the body for the data f, t and g. If, at this equilibrium configuration, the body comes in contect with the frictionless foundation, then from (2.2.12) and equations (2.3.1) - (2.3.5) it is clear that u satisfies the following system of equations and inequalities:

$$\left.\begin{aligned}
& -\sigma_{ij}(x, \nabla u)_{,j} = f_i \quad \text{in } \Omega, \\
& u_i = 0 \quad \text{on } \Gamma_D, \\
& \sigma_{ij}(x, \nabla u) n_j = t_i \quad \text{on } \Gamma_F, \\
& \sigma_{T_i}(x, \nabla u) = 0 \quad \text{on } \Gamma_c, \\
& \left.\begin{aligned}
& (u_n - g)\, \sigma_n(x, \nabla u) = 0, \\
& u_n - g \leq 0, \\
& \sigma_n(x, \nabla u) \leq 0,
\end{aligned}\right\} \quad \text{on } \Gamma_c.
\end{aligned}\right. \tag{2.3.6}$$

$\sigma_{T_i}(x, \nabla u)$ and $\sigma_n(x, \nabla u)$ denote the tangential and normal components of stress vector at a particle $x \in \Gamma_c$.

The system (2.3.6) characterizes **Signorini's problem** for the material defined by (2.3.5).

REMARK 2.3.1: Mainly we are concerned with hyperelastic Hookean materials, for which response functions in (2.3.6) take the form

$$\sigma_{ij}(x, \nabla u) = E_{ijk\ell}(x)\ u_{k,\ell}(x) \tag{2.3.7}$$

where $E_{ijk\ell}(x)$ are the components of Hooke's tensor at x and have the symmetry properties,

$$E_{ijk\ell} = E_{jik\ell} = E_{ij\ell k} = E_{k\ell ij},\ 1 \leq i,j,k,\ell \leq 3. \tag{2.3.8}$$

The Signorini's problem (2.3.6) becomes one of finding the displacement field u such that

$$\begin{aligned} &-(E_{ijk\ell}\, u_{k,\ell})_{,j} = f_i \text{ in } \Omega, \quad u_i = 0 \text{ on } \Gamma_D, \\ &E_{ijk\ell}\, u_{k,\ell}\, n_j = t_i \text{ on } \Gamma_F,\ \sigma_{T_i}(u) = 0 \text{ on } \Gamma_c, \\ &\sigma_n(u)(u_n - g) = 0,\ u_n - g \leq 0 \text{ and } \sigma_n(u) \leq 0 \text{ on } \Gamma_c. \end{aligned} \tag{2.3.9}$$

where $u_n = u.n = u_i n_i$ and $\sigma_{ij}(u) = \sigma_{ij}(x, \nabla u) = E_{ijk\ell}\, u_{k,\ell}$,

$$\sigma_n(u) = \sigma_{ij}(u) n_i n_j,\ \sigma_{T_i}(u) = \sigma_{ij}(u) n_j - \sigma_n(u) n_i,$$

$$i \leq i,j,k,\ell \leq 3. \tag{2.3.10}$$

2.4 A VARIATIONAL FORMULATION:

In the present section we give an alternative formulation of the Signorini problem (2.3.9) which plays a fundamental role in subsequent analysis.

Let V be a normed linear space of real, vector-valued, measurable functions defined on Ω. We assume that the functions v in V and their domain Ω are sufficiently smooth i.e. all operations are well defined. In particular we require that V be such that for any $u,v \in V$, the virtual work

$$\int_{\Omega} \sigma_{ij}(u)\, \varepsilon_{ij}(v)\, dx = \int_{\Omega} E_{ijk\ell}\, u_{k,\ell}\, v_{i,j}\, dx \qquad (2.4.1)$$

is well defined.

Let $v_n = v.n$ is also well defined for $v \in V$. Then the contact conditions enter the present variational formulation through the introduction of a subset K of V defined by

$$K = \left\{ v \in V : v_n - g \leq 0 \text{ on } \Gamma_c \right\}. \qquad (2.4.2)$$

Now, let u be a solution of the Signorini problem (2.3.9) and suppose that u is in V so that u is also in the set K. Let v be an arbitrary element of K and $\sigma_{ij}(u)$ is defined by (2.9.10). Then the virtual work produced by the actual stress corresponding to u and the strain $\epsilon_{ij}(v-u)$ produced by the virtual displacement $v-u$ is given by

$$\int_{\Omega} \sigma_{ij}(u) \quad \epsilon_{ij}(v-u)\,dx = \int_{\Omega} \sigma_{ij}(u)(v_i-u_i)_{,j}\,dx. \qquad (2.4.3)$$

We next assume that $\sigma_{ij}(u)$ and $(v-u)$ are such that the formula (1.2.12) is valid and hence

$$\int_{\Omega} \sigma_{ij}(u)(v_i-u_i)_{,j}dx = \int_{\Omega} -\sigma_{ij}(u)_{,j}(v_i-u_i)dx$$
$$+ \int_{\Gamma} \sigma_{ij}(u)n_j(v_i-u_i)ds.$$

Since v_i and u_i vanish on Γ_D and u is the solution of (2.3.9), for all v in K, we have

$$\int_{\Omega} \sigma_{ij}(u)(v_i-u_i)_{,j}dx = \int_{\Omega} f_i(v_i-u_i)dx + \int_{\Gamma_F} t_i(v_i-u_i)ds$$
$$+ \int_{\Gamma_c} \sigma_{ij}(u)n_j(v_i-u_i)dx.$$

Also, on Γ_c,

$$\begin{aligned}\sigma_{ij}(u)n_j(v_i-u_i) &= (\sigma_{T_i}(u) + \sigma_n(u)n_i)\ (v_i-u_i)\\ &= \sigma_{T_i}(u)(v_i-u_i) + \sigma_n(u)n_i(v_i-u_i)\\ &= \sigma_n(u)(v_n-u_n)\\ &= \sigma_n(v_n-g)\\ &> 0\end{aligned}$$

using (2.3.10).

Thus, the solution u of (2.3.9) satisfies

$$u \in K : \int_\Omega \sigma_{ij}(u)\, \epsilon_{ij}(v-u)\,dx \geq \int_\Omega f_i(v_i-u_i)\,dx + \int_{\Gamma_F} t_i(v_i-u_i)\,ds \tag{2.4.4}$$

for all v in K.

<u>REMARK</u> 2.4.1: Variational inequality (2.4.4) characterizes the solution u of the Signorini problem (2.3.9). Also (2.4.4) is known as the **primal variational principle** for the problem (2.3.9).

2.5 EXISTENCE OF SOLUTIONS OF SIGNORINI'S PROBLEM:

The present section consists in finding an equivalent variational form to the inequality (2.4.4) and the existence theorem for the new variational form. We confine ourselves to the problems defined on domains $\Omega \subset R^N$ with $N \leq 3$ in this section.

Let V and K be defined by (1.3.10) and (1.3.24) respectively. This V is the space of admissible displacements of a linearly elastic body $\bar{\Omega}$ and K is the constraint set consisting of those displacements v which satisfy the kinematical contact constraint $\gamma^{0}_{\Sigma n}(v) \leq g$ on the contact surface $\overline{\Gamma}_c \subset \Sigma$, $\gamma^{0}_{\Sigma n}$ being the normal trace operator defined in (1.3.23). Moreover K is a nonempty closed convex subset of $V \subset H^1(\Omega)$.

Let $a(.,.)$ be a bilinear form of V satisfying (1.4.5) and (1.4.6), which corresponds to the virtual work in the elastic body, that is,

$$a(u,v) = \int_{\Omega} E_{ijk\ell}\, u_{k,\ell}\, v_{i,j}\, dx, \quad u,v \in V. \tag{2.5.1}$$

Again, the elasticities $E_{ijk\ell}$ satisfy the following conditions:

(i) $E_{ijk\ell}\ L^{\infty}(\Omega)$; thus there exists a constant M such that

$$\max_{1<i,j,k,\ell<N} \| E_{ijk\ell}\|_{0,\infty} \leq M.$$

(ii) $E_{ijk\ell}(x) = E_{k\ell ij}(x) = E_{jik\ell}(x)$ almost everywhere in Ω, $1 \leq i,j,k,\ell \leq N$.

(iii) There exists a constant $m > 0$ such that a.e. in Ω,

$$E_{ijk\ell}(x)\ \epsilon_{ij}\, \epsilon_{k\ell} \geq m\ \epsilon_{ij}\, \epsilon_{ij}$$

for every $\epsilon \in R^{N\times N}$ such that $\epsilon_{ij} = \epsilon_{ji}$. (2.5.2)

Under the assumptions that

$$f \in L^2(\Omega) \quad \text{and} \quad t \in L^2(\Gamma_F),\ \Gamma_F \subset \Sigma,\ \Gamma_F \cap \Gamma_c = \phi, \tag{2.5.3}$$

the linear form

$$f(v) = \int_{\Omega} f.v dx + \int_{\Gamma_F} t.\gamma_{\Sigma}^{0}(v) ds \tag{2.5.4}$$

is bounded, that is, for all $v \in V$,

$$|f(v)| \leq C \left(\|f\|_0 + \|t\|_{0,\Gamma_F} \right) \|v\|_1 . \tag{2.5.5}$$

Then the total potential energy F is defined by

$$F(v) = \frac{1}{2} a(v,v) - f(v), \qquad v \in V. \tag{2.5.6}$$

The functional F is well defined on $H^1(\Omega)$, and therefore on K and V. Thus, the Signorini problem (2.4.4) can be given in the variational form:

$$\text{Find } u \in K : \quad a(u,v-u) \geq f(v-u), \forall v \in K \tag{2.5.7}$$

or by

$$\text{Find } u \in K : \quad F(u) \leq F(v), \forall v \in K . \tag{2.5.8}$$

The energy functional F satisfies the conditions of Theorem 1.4.3 with the help of following Lemmas:

<u>LEMMA</u> 2.5.1 [14]: Let Ω is of class $\underline{C}^{0,1}$. Then the following two conditions are equivalent:

(i) $\epsilon_{ij}(v) = 0$ in $L^2(\Omega)$, $i,j = 1,2,3$, v in $H^1(\Omega)$,

(ii) $v(x) = a+b \times x$, $a,b \in R^3$, a.e. in Ω,

where X denotes the vector product and x is the position vector of a point (x_1,x_2,x_3) in Ω.

LEMMA 2.5.2 [14]: Let Ω is of class $\underline{C}^{0,1}$. Then there exists a positive constant $C > 0$ such that

$$\int_\Omega \epsilon_{ij}(v)\ \epsilon_{ij}(v)\,dx \geq C\ \|v\|_1^2 \qquad (2.5.9)$$

for every v in V.

Hence (2.5.7) or (2.5.8) are uniquely solvable and by calling upon Theorems 1.4.2 and 1.4.4(i) we immediately arrive at the following fundamental existence theorem.

THEOREM 2.5.1 [14]: Let $F : K \longrightarrow R$ be the potential energy functional defined in (2.5.6), where K is the nonempty closed convex subset of the space V of (1.3.10). Let (2.3.4) hold and $\Omega \in \underline{C}^{0,1}$ and suppose mes $(\Gamma_D) > 0$. Then there exists a unique displacement field $u \in K$ which minimizes F on K; that is,

$$F(u) \leq F(v) \qquad \text{for all } v \in K.$$

Moreover, the minimizer u is a solution of the variational inequality,

$$a(u,v-u) \geq f(v-u) \quad \text{for all} \quad v \in K, \text{ that is,}$$

$$\int_{\Omega} E_{ijk\ell}\, u_{k,\ell}(v_{i,j}-u_{i,j})\,dx$$

$$\geq \int_{\Omega} f_i(v_i-u_i)\,dx + \int_{\Gamma_F} t_i \gamma_{\Sigma}^{0}(v_i-u_i)\,ds \qquad (2.5.10)$$

for all v in K. If all the above conditions hold but $\Gamma_D = \phi$, then a unique minimizer u of F exists in K, which is also characterized by (2.5.10), if the data f and t are such that

$$\int_{\Omega} f_i v_i\,dx + \int_{\Gamma_F} t_i \gamma_{\Sigma}^{0}(v_i)\,ds < 0$$

for all v in $K \cap R_2$, $v \neq 0$, where R_2 is defined by

$$R_2 = \left\{ v \in H^1(\Omega) : \epsilon_{ij}(v) = 0 \text{ in } L^2(\Omega), i,j = 1,2,\ldots,N \right\}$$

$$= \left\{ v \in H^1(\Omega) : v(x) = a+b \times x, \text{ a.e. in } \Omega \right\}. \qquad (2.5.11)$$

<u>REMARK</u> 2.5.1 [14]: The stress tensor σ corresponding to the displacement field u is formally defined by

$$\sigma_{ij}(u) = E_{ijk\ell}\, u_{k,\ell}$$

so the inequality (2.5.10) can also be written in the form

$$\int_{\Omega} \sigma_{ij}(u) \, \epsilon_{ij}(v-u)\,dx$$

$$\geq \int_{\Omega} f_i(v_i-u_i)\,dx + \int_{\Gamma_F} t_i(\gamma_{\Sigma}^{0}(v_i)-\gamma_{\Sigma}^{0}(u_i))\,ds \qquad (2.5.12)$$

for all v in K, where

$$\epsilon_{ij}(v-u) = \frac{1}{2}\,(v_{i,j}-u_{i,j} + v_{j,i}-u_{j,i}).$$

2.6 A FINITE ELEMENT APPROXIMATION OF SIGNORINI'S PROBLEM:

In the present section we discuss a finite element method for approximating solutions of the variational inequality (2.5.12).

Let Ω_h denote a domain, consisting of simplex elements over which each component of displacement is approximated by linear polynomials. Assembling these elements, we generate a system of piecewise linear global basis functions $\{\phi_{\alpha}\}$ such that if v_h is an approximate displacement field defined over $\bar{\Omega}_h$, then we have

$$(v_h)_i = \sum_{\alpha=1}^{P} v_i^{\alpha}\,\phi_{\alpha}(x), \qquad x \in \bar{\Omega}_h$$

and since $\phi_{\alpha}(x^{\alpha}) = \delta_{\alpha}^{\beta}$ at a node $x^{\beta} \in \bar{\Omega}_h$, $v_i^{\alpha} = (v_h)_i(x^{\alpha})$ holds.

The finite element approximation of the bilinear form a(.,.) of (2.5.1) and linear form (2.5.4) are then

$$
\begin{aligned}
a_h(u,v) &= \int_{\Omega_h} E_{ijk\ell}\, u_{k,\ell} v_{i,j}\, dx, \\
f_h(u,v) &= \int_{\Omega_h} f_i v_i\, dx + \int_{\Gamma_F^h} t_i v_i\, ds,
\end{aligned}
\tag{2.6.1}
$$

where Γ_F^h is the approximation of Γ_F. Thus, if V_h is the space spanned by $\{\phi_\alpha\}$, the evaluation of the forms (2.6.1) on V_h yields

$$
\begin{aligned}
a_h(u_h,v_h) &= E^{ik}_{\alpha\beta}\, u^\beta_k\, v^\alpha_i, \\
f_h(v_h) &= f^i_\alpha\, v^\alpha_i, \quad 1 \leq i,k \leq N,\ 1 \leq \alpha,\ \beta \leq P,
\end{aligned}
\tag{2.6.2}
$$

wherein

$$
\begin{aligned}
E^{ik}_{\alpha\beta} &= \int_{\Omega_h} E_{ijk\ell}\phi_{\beta,\ell}\, \phi_{\alpha,j}\, dx, \\
f^i_\alpha &= \int_{\Omega_h} f_i\, \phi_\alpha dx + \int_{\Gamma_F^h} t_i \phi_\alpha\, ds, \\
V_h &= \left\{ v_h \in C(\bar{\Omega}_h) : (v_h)_i = v^\alpha_i\, \phi_\alpha(x), v_h = 0 \text{ on } \Gamma_D^h \right\}.
\end{aligned}
\tag{2.6.3}
$$

The corresponding approximation of potential energy functional F of (2.5.6) is

$$
F_h(v_h) = \frac{1}{2}\, E^{ik}_{\alpha\beta}\, v^\alpha_i\, v^\beta_k - f^i_\alpha\, v^\alpha_i . \tag{2.6.4}
$$

To approximate the constraint set $K \subset V$, we use the following scheme: Let

Σ_F, Σ_D and Σ_C = the sets of all nodal points in Γ_F, Γ_D and Γ_C respectively. $v_n^{\alpha} = v_i^{\alpha} n_i^{\alpha}$, where $n_i^{\alpha} = (n_1^{\alpha}, n_2^{\alpha}, n_3^{\alpha})$ is the unit outward normal on Γ at the αth nodal point.

$$R_h = \left\{ (v_j^{\alpha}) \in R^{N-P} \; : \; v_j^{\alpha} = 0, \; 1 \leq j \leq N, \alpha \in \Sigma_D, \text{ and } v_n^{\alpha} - g^{\alpha} \leq 0, \; \alpha \in \Sigma_C \right\}.$$

Here $\quad N = \dim(\Omega)$, $\Omega \subset R^N$ and

$\quad P$ = total number of nodal points in $\bar{\Omega}_h$.

Thus, we arrive at an approximation of the constraint set K:

$$K_h = \left\{ v_h \in V_h \; : \; (v_h)_i = v_i^{\alpha} \phi_{\alpha}, \; \{v_i^{\alpha}\} \in R_h \right\}. \tag{2.6.5}$$

Then, finite element approximation of the variational inequality (2.5.12) is to find $u_h \in K_h$ such that

$$a_h(u_h, v_h - u_h) \geq f_h(v_h - u_h) \quad \text{for all} \quad v_h \in K_h \tag{2.6.6}$$

or, equivalently, find $\{u_k^{\beta}\} \in R_h$ such that

$$u_k^{\beta} \, E_{\alpha\beta}^{ik}(v_i^{\alpha} - u_i^{\alpha}) \geq f_{\alpha}^{i}(v_i^{\alpha} - u_i^{\alpha}) \quad \text{for all} \quad \{v_i^{\alpha}\} \in R_h. \tag{2.6.7}$$

Problem (2.6.6) is also equivalent to the discrete minimization problem, find $\quad u_h \in K_h : F_h(u_h) \leq F_h(v_h) \quad$ for all $\quad v_h \in K_h \qquad (2.6.8)$

where F_h is defined by (2.6.4).

Let us suppose that $\Omega_h \subset \Omega$ and let $\breve{v}_h$ denote an extension to the domain Ω of the function v_h defined on $\bar{\Omega}_h$ constructed so that the value of $\breve{v}_h$ in $\bar{\Omega}/\bar{\Omega}_h$ is defined by constant extension in the direction normal to the boundary Γ_h of Ω_h. To make clear this extension, consider the finite element on a part of the boundary of Ω_h shown in Figure 2.6.1. Within an element, a system of local coordinates (ξ, η) is defined so that the edge which coincides with the boundary is the interval (0,1) of the ξ-axis.

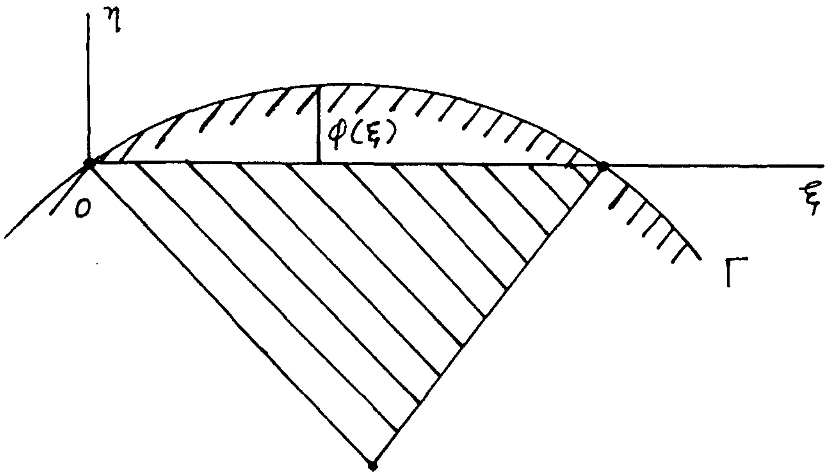

FIG. 2.6.1: Local coordinate system on edge of boundary element.

Suppose that the boundary of the domain Ω can be represented parametrically by

$$\eta = \phi(\xi) \quad \text{and} \quad \phi(\xi) \geq 0 \quad \text{for all } \xi \in (0,1) \qquad (2.6.9)$$

where $\phi(0) = \phi(1) = 0$. Then the extension $\breve{v}_h$ of v_h is defined by

$$\breve{v}_h(\xi, \eta) = v_h(\xi, 0), \; 0 \leq \eta \leq \phi(\xi).$$

Let $\breve{K}_h$ be the set of all such extensions of functions in K_h. Then, for every $\breve{v}_h$, $\breve{u}_h \in \breve{K}_h$, the following identities hold:

$$a_h(\breve{u}_h, \breve{v}_h) = a_h(u_h, v_h), \; f_h(\breve{v}_h) = f_h(v_h)$$

$$F_h(\breve{v}_h) = F_h(v_h). \qquad (2.6.10)$$

Moreover, we arrive at the following Lemma:

<u>LEMMA</u> 2.6.1 [14]: Let ϕ^I denote the linear interpolant of a function ϕ defined on the boundary Γ_C (i.e., $\phi^I(x^\alpha) = \phi(x^\alpha)$ for a node x^α on Γ_C, ϕ^I being linear between nodes). Then for every $\breve{v}_h \in \breve{K}_h$,

$$(\breve{v}_h \cdot n)^I - g^I \leq 0 \quad \text{on} \quad \Gamma_C. \qquad (2.6.11)$$

Next, we construct a function $\hat{u}_h \in \breve{K}_h$ which is the solution of the following auxiliary problem,

$$\hat{u}_h \in \breve{K}_h \; : \; a(\hat{u}_h, \breve{v}_h - \hat{u}_h) \geq f(\breve{v}_h - \hat{u}_h) \quad \text{for all } \breve{v}_h \in \breve{K}_h . \tag{2.6.12}$$

Existence and uniqueness of $\hat{u}_h$ satisfying (2.6.12) is obvious.

LEMMA 2.6.2 [14]: Let $\Omega \in \underline{C}^{1,0}$, let the bilinear form $a : V \times V \longrightarrow R$ be coercive and continuous and let

$$f \in L^2(\Omega) \quad \text{and} \quad t \in L^2(\Gamma_F) \, . \tag{2.6.13}$$

Moreover, suppose that

$$\pi^{O}_{\Sigma n}(\sigma(u)) \in L^2(\Gamma_C) . \tag{2.6.14}$$

Then, if $\hat{u}_h \in \breve{K}_h$ and $u \in K$ are solutions of (2.6.12) and (2.5.12), respectively, there exists a positive constant C_1 such that, for arbitrary $\breve{v}_h \in \breve{K}_h$,

$$C_1 \| u - \hat{u}_h \| \leq \| u - \breve{v}_h \|_1 + \left\{ \int_{\Gamma_C} |\sigma_n(u))| (|\breve{v}_{hn} - u_n| + |g - g^I| + |u^I_{hn} - u_{hn}|) ds \right\}^{1/2} , \tag{2.6.15}$$

where $\breve{v}_{hn} = \breve{v}_h . n$, $u^I_{hn} = (u_h . n)^I$, etc.

In order to construct error estimates from (2.6.15), we must determine interpolation properties of functions in the constraint set $\breve{K}_h$ instead of V_h.

LEMMA 2.6.3 [14]: Let $u \in H^2(\Omega)$ and let u^I be the unique element of the space V_h such that $u^I = u$ at all nodal points of the finite element model. Suppose that $\Omega_h \subset \Omega$ and that V_h is endowed with the usual interpolation properties. Then $\breve{u}^I \in \breve{K}_h$ and

$$\begin{aligned} \| u-\breve{u}^I\|_0 &\leq C_1 \; h^2 \| u\|_2 \; , \\ \| u-\breve{u}^I\|_1 &\leq C_2 \; h \; \| u\|_2 \; , \end{aligned} \tag{2.6.16}$$

where C_1 and C_2 are constants independent of h and u.

We next obtain an error estimate for the approximation (2.6.6).

LEMMA 2.6.4 [14]: Let the conditions of Lemma 2.6.3 and suppose that

$$u \in H^2(\Omega), \; g \in H^{3/2}(\Gamma_C), \; \Omega \in \underline{C}^{2,1} \; . \tag{2.6.17}$$

Then $C > 0$ exists such that

$$\| u-\hat{u}_h\|_1 \leq C \; (\| u\|_2, \; \| g\|_{3/2, \; \Gamma_C} \;) \; h \; . \tag{2.6.18}$$

LEMMA 2.6.5 [14]: Let $a(.,.)$ and $a_h(.,.)$ be bilinear forms defined in (2.5.1) and (2.6.1), respectively and let $f(.)$ and $f_h(.)$ be the linear forms in (2.5.4) and $(2.6.1)_2$ respectively. Suppose that $f \in L^\infty(\Omega)$ and $t \in L^\infty(\Gamma_F)$. Then there exists a positive constant C, independent of h, such that

$$\begin{aligned} |a(\breve{v}_h,\breve{w}_h) - a_h(\breve{v}_h,\breve{w}_h)| &\leq C\, h \| v_h\|_{1,\Omega_h} \|w_h\|_{1,\Omega_h} , \\ |f(\breve{v}_h) - f_h(\breve{v}_h)| &\leq C\, h \| v_h\|_{0,\Omega_h}, \end{aligned} \tag{2.6.19}$$

for all $v_h, w_h \in K_h$ where $\breve{v}_h$ and $\breve{w}_h$ are extensions of v_h and w_h into $\breve{K}_h$, respectively.

Thus, by using the results in Lemmas 2.6.1 - 2.6.5, we can conclude the following convergence theorem for the approximation of Signorini's problem (2.5.12).

THEOREM 2.6.1 [14]: Suppose that $\Omega \in \underline{C}^{2,1}$ and the solution $u \in K$ of the variational inequality (2.5.12) is regular in the sense that $u \in H^2(\Omega)$. Suppose that the function g, which characterizes the distance of the body from the rigid foundation, belongs to $H^{3/2}(\Gamma_C)$. Let $\breve{u}_h \in \breve{K}_h$ be the extension of the approximate solution $u_h \in K_h$ of (2.6.6) onto the whole domain Ω. Further, suppose that the bilinear form a(.,.) defined by (2.5.1) is coercive and continuous on V, and that the linear form f(.) defined by (2.5.4) is continuous on V. In addition, let the data $\{f,t\}$ be smooth enough, e.g., $f \in L^{\infty}(\Omega)$ and $t \in L^{\infty}(\Gamma_F)$.

Then, the sequence of extended finite element approximations $\{\tilde{u}_h\}$ converges in V to the solution $u \in K$ of the variational inequality (2.5.10) as h tends to zero. Indeed, a constant C independent of h exists such that

$$\| \tilde{u}_h - u \|_{1,\Omega_h} \leq C\, h \tag{2.6.20}$$

where h is the mesh parameter for regular refinements of the mesh.

CHAPTER-III

SIGNORINI'S PROBLEM IN LINEAR ELASTICITY WITH FRICTION

3.1 INTRODUCTION:

The Signorini problem has been discussed in the chapter II. The present chapter is devoted to the study of Signorini's problem with friction in which the normal contact pressures on the contact boundary are assumed to be prescribed.

One physical situation which is roughly simulated by the special conditions is when a thin belt or tape is pressed against an elastic body by prescribed pressures and the belt or tape simultaneously resists tangential motion by the development of frictional stresses on the contact surface. σ_n is then given on Γ_C but tangential stress σ_T due to friction are unknown. While this particular problem may at first seem to be a bit contrived, infact, this special case is the key to many important results pertaining to the general contact problem with friction. Indeed, the analysis of problems with prescribed tangential stress can be combined to yield informatioı on the behaviour of finite element approximations of contact problems in elastostatics with Coulomb friction.

The chapter has been divided into five sections. In Section 3.2, Signorini's problem has been studied. Section 3.3, deals with the variational formulation of the problem. Section 3.4 consists existence theorem of the variational problem obtained by placing restrictions on the contact stresses. In Section 3.5 we have discussed the regularization of the Signorini problem with friction and then finite element approximations of problems in Section 3.4 and 3.5 have been discussed in Section 3.6.

The results mentioned in this chapter are mainly due to Campos, Oden and Kikuchi [06],Oden and Campos [31] and Kikuchi and Oden [14].

3.2 SIGNORINI'S PROBLEM WITH COULOMB FRICTION:

Let $f = (f_1, f_2, f_3)$ and $t = (t_1, t_2, t_3)$ be a body force and a traction vector, respectively, on a part of the boundary Γ_F, of the body. Let the body be fixed on a part of the boundary, Γ_D. Then the Signorini problem of a linearly elastic body with Coulomb friction can be represented by the following boundary value problem:

$$\sigma_{ij}(u)_{,j} + f_i = 0, \quad \sigma_{ij}(u) = E_{ijk\ell}u_{k,\ell} \text{ in } \Omega,$$

$$u_i = 0 \text{ on } \Gamma_D, \ \sigma_{ij}(u)n_j = t_i \text{ on } \Gamma_F, \qquad (3.2.1a)$$

$$\sigma_n(u) = 0 \text{ and } \sigma_T(u) = 0 \text{ if } u_n < g \text{ on } \Gamma_C,$$

and

if $u_n = g$ on Γ_C, then $\sigma_n(u) < 0$ and

if, in addition, $|\sigma_T(u)| < \nu_F \ |\sigma_n(u)|$ then $u_T = 0$ while

if $|\sigma_T(u)| = \nu_F \ |\sigma_n(u)|$ then there exists $\lambda \geq 0$

such that $u_T = - \lambda \, \sigma_T(u)$. (3.2.1b)

Here, (3.2.1b) define a form of Coulomb's law of friction for elastostatic problems, ν_F is the coefficient of friction, $\nu_F \in L^\infty(\Gamma_C)$, $\nu_F \geq \nu_0 > 0$ a.e. on Γ_C.

<u>REMARK</u> 3.2.1: Conditions (3.2.1b) can also be written in the equivalent, perhaps more familiar forms:

$$|\sigma_T(u)| < \nu_F|\sigma_n(u)|, \ (\nu_F|\sigma_n(u)| - |\sigma_T(u)|)u_T = 0 \qquad (3.2.2a)$$

$$|\sigma_T(u)| \leq \nu_F|\sigma_n(u)|, \ \sigma_T \cdot u_T + \nu_F|\sigma_n(u)||u_T| = 0. \qquad (3.2.2b)$$

REMARK 3.2.2: Throughout this chapter we assume that conditions (2.5.2) on elasticities hold and

$$\begin{aligned} &\Omega \subset R^N \quad \text{and} \quad \partial\Omega \text{ is sufficiently smooth} \\ &f_i \in L^2(\Omega), \quad t_i \in L^2(\Gamma_F), \quad \bar{\Gamma}_C \cap \bar{\Gamma}_D = \phi \ . \end{aligned} \tag{3.2.3}$$

3.3 VARIATIONAL FORMULATION OF THE PROBLEM:

In the present section we derive a variational formulation of the problem (3.2.1).

We start by introducing some familiar notations:

$$V = \left\{ v \in H^1(\Omega) \ : \ \gamma_D(v) = 0 \text{ on } \Gamma_D \right\},$$

$$K = \left\{ v \in V \ : \ \gamma^0_{\Sigma n}(v) - g \leq 0 \ \text{ on } \ \Gamma_C \ \text{ in } \ H^{1/2}(\Gamma_C), \quad \Sigma = \text{int}\,(\Gamma - \Gamma_D), \ \Gamma_C \subset \Sigma \right\},$$

$$\begin{aligned} a(u,v) &= \int_\Omega \sigma_{ij}(u)\, \epsilon_{ij}(v-u)\,dx, \quad u,v \in V, \\ f(v) &= \int_\Omega f.v\,dx + \int_{\Gamma_F} t.v\,ds, \quad v \in V \ . \end{aligned} \tag{3.3.1}$$

In addition, we introduce a nonlinear functional $j(.,.)$ which will serve to characterize the virtual work of the frictional forces

$$\begin{aligned} &j \ : V \times V \longrightarrow R, \\ &j(u,v) = \int_\Gamma \nu_F |\sigma_n(u)| \ |v_T|\,ds. \end{aligned} \tag{3.3.2}$$

Here $\sigma_n(u) = \sigma_{ij}(u)n_i n_j = E_{ijk\ell}u_{k,\ell}\, n_i n_j$ is the normal stress developed on Γ_C and v_T is the tangential component of the displacement v on Γ_C. The functional $v \longrightarrow j(v,v)$ is a nonconvex, nondifferentiable and nonquadratic functional of the admissible displacements v in K.

For simplicity we use the following conventional notations:

$$\pi_T(\sigma) = \sigma_T,\ \pi_n(\sigma) = \sigma_n,\ \gamma^0_{\Sigma T}(u) = u_T$$

$$\gamma^0_{\Sigma n}(u) = u_n \quad \text{etc.}$$

With the above preparations in place, we consider the general variational principle for the Signorini problem (3.2.1) with Coulomb friction:

$$u \in K: \quad a(u,v-u)+j(u,v)-j(u,u) \geq f(v-u) \quad \forall v \in K \cdot \qquad (3.3.3)$$

<u>THEOREM</u> 3.3.1 [14]: If u is a sufficiently smooth solution of (3.3.3) then u satisfies (3.2.1). Conversely, any solution of (3.2.1) satisfies (3.3.3).

<u>REMARK</u> 3.3.1: The more general form of inequality (3.3.3) has been studied by Noor [28] and Siddiqi and Ansari [37].

3.4 EXISTENCE OF SOLUTIONS OF THE PROBLEM:

The present section consists in finding a reduced variational inequality to the inequality (3.3.3) and existence theorem to the reduced model. The question of existence of solutions to the general problem (3.3.3) is still an open problem.

We begin with the elastostatic contact problem (3.2.1) in which the normal stress $\sigma_n(u)$ is prescribed on Γ_C. In this case, Γ_C is known in advance, infact is a part of Γ_F and the unilateral constraint may or may not be imposed. For simplicity, we shall not impose this constraint for the present, so that points on Γ_C can display freely in a direction normal to the boundary upon the application of loads. However, tangential displacements u_T of points on Γ_C are resisted in accordance with a Coulomb friction law of the type (3.2.1b).

The classical statement of this restricted class of friction problem is to find a sufficiently smooth displacement field u such that

$$
\begin{aligned}
&(E_{ijkl}u_{k,l})_{,j} + f_i = 0 \quad \text{in } \Omega, \\
&u_i = 0 \quad \text{on} \quad \Gamma_D, \quad \nu_F|\sigma_n(u)| = s \quad (s > 0), \\
&|\sigma_T| < s \Longrightarrow u_T = 0, \\
&|\sigma_T| = s \Longrightarrow \text{there exists } \lambda \geq 0 \quad \text{on } \Gamma_C \\
&\qquad\qquad \text{such that} \quad u_T = -\lambda\sigma_T
\end{aligned}
\tag{3.4.1}
$$

where s is regarded as given. In this case, the friction functional $j : V \times V \longrightarrow R$ of (3.3.2) reduces to a functional $j : V \longrightarrow R$ defined by

$$
j(v) = \int_{\Gamma_F} s\ |v_T|\ ds. \tag{3.4.2}
$$

In addition, the linear functional f of (3.3.1), is now replaced by the linear form

$$
f(v) = \int_{\Omega} f.v dx - \int_{\Gamma_F} sv_n ds \tag{3.4.3}
$$

where we assume that $f_i \in L^2(\Omega)$ and, for instance, $s \in L^2(\Gamma_F)$. The variational boundary value problem (3.3.3) now reduces to

$$
u \in V : \quad a(u, v-u) + j(v) - j(u) \geq f(v-u) \quad \forall v \in V. \tag{3.4.4}
$$

Now, we give an existence theorem for the problem (3.4.4) as follows:

THEOREM 3.4.1 [14]: If either mes $\Gamma_D > 0$ or $\Gamma_D = \phi$ and the result $j(r) - |f(r)| \geq C \| r\|_0$, $r \in R_2$ of (2.5.11), holds, then there exists at least one minimizer u in V of the functional F defined by

$$F : V \longrightarrow R,$$

$$F(v) = \frac{1}{2} a(v,v) - f(v) + j(v)$$

is weakly lower semicontinuous on all of V. Moreover, each minimizer u is a solution of the variational inequality (3.4.4).

REMARK 3.4.1: The abstract form of (3.4.4) has also been studied by Glowinski, Lions and Trémolieres [11].

3.5 A REGULARIZED PROBLEM:

A complicating aspect of the reduced boundary value problem (3.4.4) is that the friction functional j is non-differentiable. This property is especially troublesome when one attempts to develop numerical schemes for the analyses of problems of this type. On the other hand, the fact that $j(.)$ is not Gâteaux differentiable at the origin is not too surprising because the Euclidean norm $|-|$ in R^N is also not differentiable at the origin nor is the function $x \longrightarrow |x|$ in R^1.

We note that the approximation of such non-differentiable functionals $j(\cdot)$ by differentiable regularized functionals $j\ (\cdot)$ that depend upon a real parameter $\epsilon > 0$ has been used by Moreau and Brezis in the study of optimization problems and by Glowinski, Lions and Trémolieres, Campos, Oden and Kikuchi [06] and Kikuchi and Oden [14] in the development of numerical methods for the solution of variational inequalities of the type (3.4.4).

Towards the construction of a regularization of $j(\cdot)$, we first consider smooth approximations of the function $x \longrightarrow |x|$. As example, consider the function

$$\phi_\epsilon(x) = \begin{cases} x - \frac{1}{2}\epsilon , \\ \frac{1}{2\epsilon}\, x^2 , \\ -\ x + \frac{1}{2}\epsilon , \end{cases} \qquad \frac{d}{dx}\,\phi(x) = \begin{cases} 1 & \text{if } x \geq \epsilon , \\ \frac{1}{\epsilon}\, x & \text{if } |x| < \epsilon , \\ -1 & \text{if } |x| \leq -\epsilon . \end{cases} \tag{3.5.1}$$

It is easily varified that this approximation converges in R to the absolute value functions as $\epsilon \longrightarrow 0$. Approximation is shown in the Figure 3.5.1, exhibits a piecewise linear first derivative.

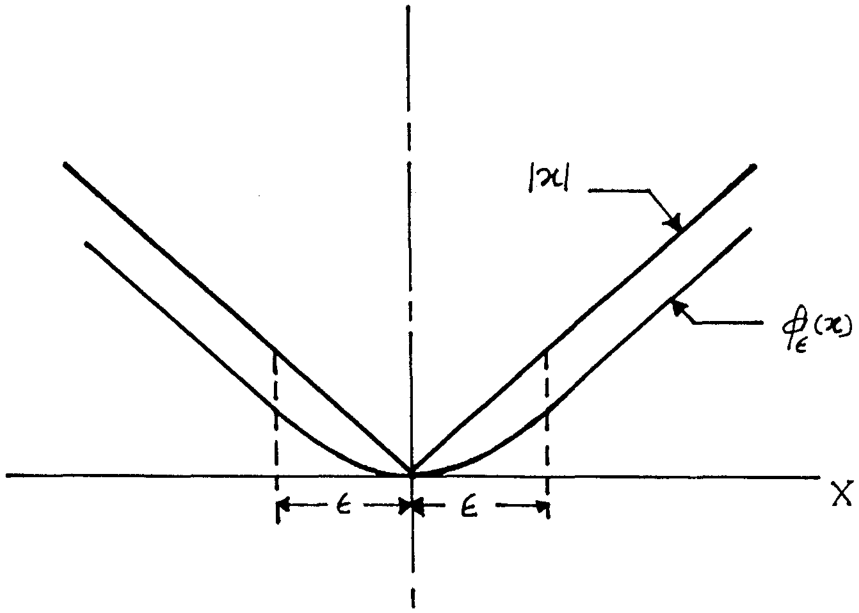

FIG. 3.5.1: Regularization of $|x|$ by a differentiable approximation ϕ (x).

Note that the generalization of (3.5.1) to vectors $x \in R^n$ is immediate. If $x = (x_1, x_2, \ldots\ldots, x_n) \in R^n$ and $|x| = \sqrt{x.x}$, then we set

$$\phi_\epsilon(x) = \begin{cases} |x| - \epsilon/2 & \text{if } |x| > \epsilon \\ \frac{1}{2\epsilon}\, x.x & \text{if } |x| \leq \epsilon\,. \end{cases} \qquad (3.5.2)$$

Thus, for a vector $v \in V$, we define $\phi_\epsilon : V \longrightarrow L^1(\Gamma_C)$ by

$$\phi_\epsilon(v) = \begin{cases} |v_T| - \epsilon/2 & \text{if } |v_T| > \epsilon\,, \\ \frac{1}{2\epsilon}\, |v_T|^2 & \text{if } |v_T| < \epsilon\,, \end{cases} \qquad (3.5.3)$$

wherein v_T is understood to mean the tangential component of the trace $\gamma(v)$ of v on the portion Γ_C of the boundary. We can easily varify that $v \longrightarrow \phi_\epsilon(v)$ is convex, continuous and that $[0,1] \longrightarrow \phi_\epsilon(u+\theta v)$ is $C^1[0,1]$ for any $u,v \in V$.

As an approximation of the friction functional j, we introduce

$$j_\epsilon : V \longrightarrow R, \quad j_\epsilon(v) = \int_{\Gamma_C} s\phi_\epsilon(v)ds \tag{3.5.4}$$

where now we assume $s \in L^2(\Gamma_C)$, $s \geq 0$ a.e. on Γ_C. The major properties of j_ϵ are summarized in the following lemma:

<u>**LEMMA**</u> 3.5.1 [14]: The functional $j_\epsilon : V \longrightarrow R$ defined by (3.5.4) is convex and Gâteaux differentiable and, hence weakly lower semicontinuous on all of V for every $\epsilon > 0$. Moreover, its Gâteaux derivative is given by

$$\langle Dj_\epsilon(u),v\rangle = \lim_{\theta \longrightarrow 0^+} \int_{\Gamma_C} s \frac{\partial}{\partial\theta} \phi_\epsilon(u+\theta v)ds \quad \forall u,v \in V. \tag{3.5.5}$$

In addition,

$$|j(v)-j_\epsilon(v)| \leq (2 \| s \|_{0,1,\Gamma_C})\epsilon \ \text{mes}\ \Gamma_C \tag{3.5.6}$$

for all $v \in V$ so that $j_\epsilon(v) \longrightarrow j(v)$ as $\epsilon \longrightarrow 0$.

Now, we turn to the regularized total potential energy functional $F_\epsilon : V \longrightarrow R$ given by

$$F_\epsilon(v) = \frac{1}{2} a(v,v) - f(v) + j_\epsilon(v). \tag{3.5.7}$$

This functional is strictly convex and Gâteaux differentiable obviously. It is coercive on V for mes $\Gamma_D > 0$.

<u>THEOREM</u> 3.5.1 [14]: Let conditions (3.2.3) hold, $s \in L^\infty(\Gamma_C)$, and mes $\Gamma_D > 0$. Then, for any $\epsilon > 0$, there exists a unique minimizer $u_\epsilon \in V$ of the perturbed potential energy functional F_ϵ of (3.5.7). Moreover, u_ϵ is characterized as the solution of the variational equality,

$$a(u_\epsilon, v) + \langle Dj_\epsilon(u_\epsilon), v\rangle = f(v) \tag{3.5.8}$$

for all v in V.

Now, we give a result that shows u_ϵ converges strongly to u in V and also provide an estimate of the rate of convergence.

<u>THEOREM</u> 3.5.2 [14]: Let u and u_ϵ be solutions of (3.4.4) and (3.5.8), respectively. Then there exists a constant C independent of ϵ such that

$$\| u-u_\epsilon \|_1 \leq C \sqrt{\epsilon} \, . \tag{3.5.9}$$

Thus, $u_\epsilon \longrightarrow u$ strongly in V as $\epsilon \longrightarrow 0$.

3.6 FINITE ELEMENT APPROXIMATIONS:

In this section we discuss finite element approximations of problem (3.4.4) or its regularization (3.5.8).

We construct a family of finite dimensional subspaces V_h of V by approximating the test functions v_i by piece-wise polynomials over a partition Ω_h of Ω consisting of $E = E(h)$ finite elements. Suppose that the spaces V_h exhibit the following interpolation properties:

Given $v \in V = (H^m(\Omega))^n$, $m > 0$, and supposing that V_h is generated by local element shape functions that contain complete polynomials of degree $\leq k$, there exists a constant C, independent of v and the mesh size h, and a vector field $v_h \in V_h$ such that

$$\| v-v_h \|_s \leq C \, h^\mu \| v \|_m \, ,$$

$$\mu = \min\,(k+1 - s,\ m-s), \quad s = 0,1. \tag{3.6.1}$$

In addition, for $p \in R$, there exists a constant C such that

$$\| \gamma(v) - \gamma(v_h) \|_{p,\Gamma_C} \le C_1 \, h^{r-p} \, \|\gamma(v)\|_{r,\Gamma_C}$$

whenever $\gamma(v) \in H^r(\Gamma_C)$. (3.6.2)

The finite element approximation (3.4.4) is of the form

$$u_h \in V_h : \quad a(u_h, v_h - u_h) + j(v_h) - j(u_h) \ge f(v_h - u_h) \qquad (3.6.3)$$

for all v_h in V_h, whereas the finite element approximation of the regularized problem (3.5.8) is characterized by

$$u_h^\epsilon \in V_h : \quad a(u_h^\epsilon, v_h) + \langle Dj_\epsilon(u_h^\epsilon), v_h \rangle = f(v_h) \qquad (3.6.4)$$

for all v_h in V_h.

Moreover, under the assumptions of Theorems 3.4.1 and 3.5.1, there exist unique solutions u_h and u_h^ϵ to (3.6.3) and (3.6.4) respectively, the latter existing for each $\epsilon > 0$. Thus for fixed h, a constant C exists such that

$$\| u_h - h_u^\epsilon \|_1 \le C \sqrt{\epsilon} \; . \qquad (3.6.5)$$

We need the following intermediate result to establish the error estimate.

THEOREM 3.6.1 [14]: Let u denote the solution of (3.4.4) and suppose that

$$u \in H^{r}(\Omega), \ \gamma(u)|_{\Gamma_C} \in H^{r-1/2}(\Gamma_C),$$

$$\sigma_T(u) \in H^{r-3/2}(\Gamma_C) \quad \text{with} \quad s \in H^{r-3/2}(\Gamma_C).$$

In addition, let u_h be the solution of (3.6.3) and suppose that V_h is such that the interpolation properties (3.6.1) and (3.6.2) hold. Then

$$\|u-u_h\|_1 \leq C\,(\|u\|_r)\ h^{r-1} \tag{3.6.6}$$

where C is a constant independent of h.

The final error estimate is now an immediate consequence of (3.6.5), (3.6.6) and the triangle inequality:

THEOREM 3.6.2 [14]: Let (3.6.5) and the conditions of Theorem 3.6.1 hold. Then the error in the approximation of (3.4.4) and (3.6.4) satisfies

$$\|u-u_h^{\epsilon}\|_1 \leq C\,(\sqrt{\epsilon} + h^{r-1}).$$

In particular, if $r = 2$,

$$\|u-u_h^{\epsilon}\|_1 = O\,(\sqrt{\epsilon} + h).$$

CHAPTER-IV

RIGID PUNCH PROBLEM OF A LINEARLY ELASTIC INCOMPRESSIBLE BODY

4.1 INTRODUCTION:

A contact problem of incompressible linearly elastic material is concerned in this chapter. The problem is to find a deformed configuration of an elastic foundation (body) indented by a rigid punch using the finite element method. The formulation of the rigid punch problem is given by variational inequalities together with existence and uniqueness theorems of the solution. An interesting point is that not only forces but also moments are applied on the rigid punch. Thus, the rigid punch rotates as well as being indented into the foundation.

The formulation of a rigid punch problem by variational inequalities was given by Duvant, where only forces are applied on the rigid punch. The present study involves not only forces but also moments. This requires some what delicate arguments on existence of the solution. The compatibility condition on the applied forces and moments must be introduced to prove coerciveness of the total potential energy, or equivalently, of the virtual work. It should be noted that frictional effects on the contact surface are completely neglected in this chapter.

The contact condition for arbitrary shapes of surfaces of the rigid punch and elastic body is obtained in this chapter, whereas it is usually neglected. A linearization is applied to obtain a well defined contact condition from the original nonlinear nonconvex type constraint. Details of the derivation of the contact condition can be found in the monograph by Kikuchi and Oden [14]. One references therein together with mathematical arguments, such as variational formulations, existence and uniqueness theorems, error estimates for finite element approximations, solution methods, etc. to various contact problems.

Most of the results in this chapter, are due to Kikuchi and Song [15].

4.2 CONTACT CONDITIONS:

In this section we discuss kinematical and stress contact conditions for the plane problems, since its generalization to the three dimensional setting seems to be straightforward.

Let the surface of the rigid punch be given by

$$x_2 = \psi(x_1) \tag{4.2.1}$$

and let the surface of an elastic body Ω be given by

$$x_2 = \phi(x_1) \tag{4.2.2}$$

The conditions (4.2.1) and (4.2.2) of the surfaces are restricted only on the possible contact surfaces. Suppose $\psi(0) = \phi(0) = 0$. For a moment, ϕ and ψ are assumed to be infinitely differentiable.

Let the rigid punch be indented into the body and let it be rotated around the origin of the coordinate system. The depth of indentation is denoted by α and the rotation of the punch is represented by θ. Then the kinematical contact condition is given by

$$
\begin{aligned}
&x_1+u_1(x_1,\phi(x_1) = \bar{x}_1 \operatorname{Cos}\theta - \psi(\bar{x}_1) \operatorname{Sin}\theta \\
&\phi(x_1)+u_2(x_1,\phi(x_1)) \leq \alpha+\bar{x}_1 \operatorname{Sin}\theta + \psi(\bar{x}_1) \operatorname{Cos}\theta
\end{aligned}
\tag{4.2.3}
$$

where $u = (u_1, u_2)$ is the displacement field of the elastic body, and $\{x_1, \bar{x}_1\}$ is the pair of the initial coordinates of the particles of the body and the punch which occupy the same place at the deformed configuration. Now, we assume that

(i) If θ is small enough then $\operatorname{Sin}\theta = \theta$ and $\operatorname{Cos}\theta = 1$,

(ii) If $u_1+\psi\theta$ is small enough then the higher order terms of the Taylor expansion of ψ at x_1 are negligible.

(4.2.4)

Under these assumptions it follows from (4.2.3) that

$$\phi(x_1)+u_2(x_1,\phi(x_1)) \leq \alpha+x_1\theta+\psi(x_1)+\psi'(x_1)(u+\psi\theta).$$

Here $\psi' = \frac{\partial \psi}{\partial x_1}$. Dividing by $(1+(\psi'(x_1))^2)^{1/2}$ and noting that the inward unit vector N to the surface of the rigid punch is given by $N = \text{grad}(-\psi+x_2)/|\text{grad}(-\psi+x_2)|$, the following compact form of the kinematical restriction is obtained:

$$u_N(x_1) \leq ((\alpha+s)N_2+(x_1N_2-\psi N_1)\theta)(x_1), \tag{4.2.5}$$

where $u_N = u_1N_1 + u_2N_2$, $N = (N_1,N_2)$, and $s = \psi-\phi$. That is, the condition (4.2.4) must be held on the possible contact surface Γ_C of the elastic body. It is notable that $N_i(x_1)$ is the ith component of the inward normal unit vector on the rigid punch evaluated at the coordinate x_1 which indicates the particle of the elastic body on Γ_C. For simplicity (4.2.5) is merely written as

$$u_N \leq (\alpha+s)N_2 + (x_1N_2 - \psi N_1)\theta$$

without the coordinate x_1.

Next we discuss stress conditions. It is certain that

$$\begin{aligned} \sigma_N \leq 0 \quad &\text{if} \quad u_N = (\alpha+s)N_2 + (x_1N_2 - \psi N_1)\theta \\ \sigma_N = 0 \quad &\text{if} \quad u_N < (\alpha+s)N_2 + (x_1N_2 - \psi N_1)\theta \end{aligned}, \tag{4.2.6}$$

where σ_N is the normal stress of the traction vector induced by the stress tensor with respect to the normal vector N on the rigid punch. The traction vector σ is defined by

$$\sigma_i = \sigma_{ij} n_j \quad \text{on} \quad \Gamma_C$$

and decomposed by

$$\sigma = \sigma_N N = \sigma_T$$

where $n = (n_1, n_2)$ is the outward normal unit vector on the boundary of the elastic body, and σ_T is the tangential component of the traction to the surface of the rigid punch. Similarly, the displacement u is also decomposed by

$$u = u_N N + u_T.$$

If surface of the rigid punch and the elastic body is lubricated so that frictional effects may be neglected, the condition

$$\sigma_T = 0 \tag{4.2.7}$$

is also imposed. Combining (4.2.5) - (4.2.7), contact conditions are summarized by

$$\begin{aligned} \sigma_N(u_N-(\alpha+s)N_2-(x_1N_2-\psi N_1)\theta) &= 0, \\ u_N-(\alpha+s)N_2-(x_1N_2-\psi N_1)\theta &\leq 0, \\ \sigma_T = 0,\ \sigma_N &\leq 0. \end{aligned} \tag{4.2.8}$$

If the rigid punch is indented and rotated, the balance of forces and moments must be satisfied on the contact surface. These imply the balance equations of the force and moment

$$\int_{\Gamma_C} \sigma_N N_2 \, dx_1 - P = 0,$$

$$\int_{\Gamma_C} (\sigma_N N_2 x_1 - \sigma_N N_1 \psi) dx_1 - M = 0, \tag{4.2.9}$$

where P and M are the applied force and moment on the rigid punch.

<u>REMARK</u> 4.2.1: If the linearization is performed with respect to the coordinate x_1 of the rigid punch, the kinematical restriction (4.2.5) is written by

$$u_n(x_1) \leq (\alpha + s(x_1)) n_2(x_1) + (x_1 n_2(x_1) - \psi(x_1) n_1(x_1))\theta \tag{4.2.10}$$

where $u_n = u.n$, and $n = (n_1, n_2)$ is the outward normal unit vector on the boundary of the elastic body. Then the conditions (4.2.8) and (4.2.9) are replaced by

$$\sigma_n(u_n - (\alpha+s)n_2 - (x_1 n_2 - \psi n_1)\theta) = 0,$$

$$u_n - (\alpha+s)n_2 - (x_1 n_2 - \psi n_1)\theta \leq 0,$$

$$\sigma_T = 0, \quad \sigma_n \leq 0. \tag{4.2.11}$$

and

$$\int_{\Gamma_C} \sigma_n n_2 x_1 - P = 0,$$

$$\int_{\Gamma_C} (\sigma_n n_2 x_1 - \sigma_n n_1 \psi)\, dx_1 - M = 0. \qquad (4.2.12)$$

For the details of these derivations see Kikuch and Oden [14].

4.3 INCOMPRESSIBILITY:

Only small strain cases are considered in this section. Then incompressible condition of a material is given by

$$\text{div } u = 0 \quad \text{in } \Omega \qquad (4.3.1)$$

where u is the displacement vector. The condition (4.3.1) is also represented by

$$\epsilon_{ij}(u) = \epsilon_{11}(u) + \epsilon_{22}(u) = 0 \qquad (4.3.2)$$

where $\epsilon_{ij}(u)$ is the strain tensor defined by

$$\epsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

is the sense of distribution. Under this constraint, the total strain energy of an isotropic material is given by

$$W(u) = \int_{\Omega} \mu\, \epsilon_{ij}(u)\, \epsilon_{ij}(u)\, dx. \qquad (4.3.3)$$

The constitutive relation is then described by the equation

$$\sigma_{ij}(u) = -p\,\delta_{ij} + 2\mu\,\epsilon_{ij}(u) \tag{4.3.4}$$

where p is the pressure field, which may be identified with Lagrange multiplier to the incompressibility $\operatorname{div} u = 0$ in Ω.

4.4 RIGID PUNCH PROBLEM

Let $f = (f_1, f_2)$ be a body force and let $t = (t_1, t_2)$ be a traction vector on a part of the boundary, Γ_F, of the body. Let the body be fixed on a part of the boundary, Γ_D. Then, the rigid punch problem of an incompressible linearly elastic body can be represented by the following boundary value problem:

$$\left.\begin{aligned} -\sigma_{ij}(u)_{,j} &= f_i \\ \sigma_{ij}(u) &= -p\,\delta_{ij} + 2\mu\,\epsilon_{ij}(u), \\ &\text{and } \operatorname{div} u = 0 \end{aligned}\right\} \text{in } \Omega \tag{4.4.1}$$

$$u = 0 \quad \text{on } \Gamma_D, \tag{4.4.2}$$

$$\sigma_{ij}(u)n_j = t_i \quad \text{on } \Gamma_F, \tag{4.4.3}$$

$$\left.\begin{aligned} \sigma_N(u)(u_N-(\alpha+s)N_2-(x_1N_2-\psi N_1)\theta) &= 0 \\ u_N-(\alpha+s)N_2-(x_1N_2-\psi N_1)\theta &\le 0 \\ \sigma_T(u) = 0,\ \sigma_N &\le 0 \end{aligned}\right\} \text{on } \Gamma_C \tag{4.4.4}$$

$$\int_{\Gamma_C} \sigma_N \, N_2 dx_1 - P = 0 \tag{4.4.5}$$

$$\int_{\Gamma_C} (\sigma_N \, N_2 x_1 - \sigma_N N_1 \psi) dx_1 - M = 0. \tag{4.4.6}$$

Here, pointwise existence of the stress and its divergence, decomposition of the traction, and integrability of the normal stress are assumed. In the variational formulation these conditions are weakend.

<u>REMARK</u> 4.4.1: If the contact conditions (4.2.11) and (4.2.12) are applied, (4.4.4) - (4.4.6) are replaced by (4.2.11) and (4.2.12).

<u>REMARK</u> 4.4.2: No body force and no traction on Γ_F are assumed in the subsequent discussions of this chapter, i.e.

$$f = 0, \quad t = 0. \tag{4.4.7}$$

4.5 <u>VARIATIONAL FORMULATION AND EXISTENCE THEOREM</u>:

In this section we will obtain the variational formulation of the rigid punch problem given in Section 4.4. Existence and uniqueness of the solution of the variational form will also be discussed.

Suppose that the triplet (u,α,θ) satisfies the boundary value problem (4.4.1) - (4.4.6). Let (v,β,δ) be an arbitrary triplet such that $v = 0$ on Γ_D, $(\beta,\delta) \in R \times R$, and

$$v_N \leq (\beta+s)\, N_2 + (x_1 N_2 - \psi N_1)\delta \text{ on } \Gamma_C ,$$

$$\text{div } v = 0 \text{ in } \Omega .$$

Then

$$\int_\Omega 2\,\mu\, \epsilon_{ij}(u)\, \epsilon_{ij}(v-u)dx$$

$$= \int_\Omega (-\sigma_{ij}(u)_{,j})(v-u)_i dx + \int_\Gamma \sigma_{ij}(u) n_j (v-u)_i ds$$

$$= \int_\Gamma \sigma_{ij}(u) n_j (v-u) ds.$$

Applying the inequality

$$\sigma_{ij}(u) n_j (v-u)_i \geq \sigma_N(u) N_2(\beta-\alpha) + \sigma_N(u)(x_1 N_2 - \psi N_1)(\delta-\theta),$$

it can be obtained that

$$\int_\Omega 2\,\mu\, \epsilon_{ij}(u)\, \epsilon_{ij}(v-u)dx \geq P(\beta-\alpha) + M(\delta-\theta). \qquad (4.5.1)$$

Set

$$a(u,v) = \int_\Omega 2\,\mu\, \epsilon_{ij}(u)\, \epsilon_{ij}(v)dx . \qquad (4.5.2)$$

Then, if there exist positive constants C_1 and C_2 such that

$$C_1 \leq \mu(x) \leq C_2 \quad \text{a.e. in } \Omega\,, \tag{4.5.3}$$

Korn's inequalities (1.3.25) and (1.3.26) imply existence of positive constants m_1 and m_2 such that

$$\begin{aligned} a(u,u) &\geq m_1 \|u\|_1^2 \,, \\ a(u,v) &\leq m_2 \|u\|_1 \|v\|_1 \,, \end{aligned} \tag{4.5.4}$$

for every $u,v \in V_D$, where

$$V_D = \left\{ v \in H^1(\Omega) \;:\; v = 0 \quad \text{a.e.} \quad \text{on} \quad \Gamma_D \right\} \tag{4.5.5}$$

and mes $(\Gamma_D) > 0$.

Let the set C be defined by

$$\begin{aligned} C = \Big\{ (v,\beta,\delta) \in V_D \times R \times R \;:\; \operatorname{div} v = 0 \quad \text{a.e.} \quad \text{in } \Omega \\ \text{and} \quad v_N \leq (\beta+s)N_2+(x_1N_2-\psi N_1)\delta \ \text{a.e. on} \ \Gamma_C \Big\}. \end{aligned} \tag{4.5.6}$$

Since the trace operator from $H^1(\Omega)$ into $H^{1/2}(\Gamma)$ is continuous, and since divergence operator $\operatorname{div} : H^1(\Omega) \longrightarrow L^2(\Omega$ is continuous, the set C is closed in $H^1(\Omega)$. Convexity of C follows from linearity of the trace and divergence operators. Since $H^1_{0,\operatorname{div}}(\Omega) \subset V_D$, the set C is nonempty. Here

$$H^1_{0,\mathrm{div}}(\Omega) = \{v \in H^1_0(\Omega) : \mathrm{div}\ v = 0 \ \text{ in } \Omega\}. \qquad (4.5.7)$$

Using the set C and the bilinear form $a(.,.)$, (4.5.1) can be written by

$$(u,\alpha,\theta) \in C \ : \ a(u,v-u) \geq P(\beta-\alpha) + M\,(\delta-\theta)$$

$$\text{for every } (v,\beta,\delta) \in C. \qquad (4.5.8)$$

Since the bilinear form $a(.,.)$ is symmetric, the variational inequality (4.5.8) is equivalently represented by the constrained minimization problem

$$(u,\alpha,\theta) \in C \ : \ F(u,\alpha,\theta) \leq F(v,\beta,\delta) \qquad (4.5.9)$$

where

$$F(v,\beta,\delta) = \frac{1}{2}\, a(v,v) - P\beta - M\delta \,. \qquad (4.5.10)$$

Thus, the boundary value problem (4.4.1) - (4.4.6) is reduced to the variational problem (4.5.8), or equivalently (4.5.9).

<u>THEOREM</u> 4.5.1 [15]: (Existence and Uniqueness). Let the domain Ω be Lipschitzian, and let the surface of the rigid punch is continuously differentiable piecewise on Γ_C. Suppose that (4.5.3) holds, and that the following compatibility condition of the applied force and moment holds:

There exist points $(d_1,\phi(d_1))$ and $(d_2,\phi(d_2))$ on the boundary Γ_C such that

$$P(d_2-\Psi(d_2)N_1(d_2)/N_2(d_2))\leq M\leq P(d_1-\Psi(d_1)N_1(d_1)/N_2(d_2)) \tag{4.5.11}$$

and

$$P < 0 \quad \text{and} \quad N_2 > 0; \tag{4.5.12}$$

Then there exists a unique solution $(u,\alpha,\theta)\in C$ to the variational problem (4.5.8), or equivalently (4.5.9).

<u>REMARK</u> 4.5.1: If the contact conditions (4.2.11) and (4.2.12) are adopted instead of (4.2.8) and (4.2.9), the admissible set C must be changed by

$$C^* = \Big\{(v,\beta,\delta)\in V_D \times R \times R : \text{div } v = 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad v_n \leq (\beta+s)n_2+(x_1n_2-\Psi n_1) \quad \text{a.e. on } \Gamma_C\Big\} \tag{4.5.13}$$

where the variational inequality (4.5.8) is still valid.

To ensure existence of a solution, some sort of smoothness of the surface Γ_C of the body must be assumed. That is, $\phi(.)$ is to be continuously differentiable, piecewise on Γ_C.

The vector N must be changed by n in the compatibility condition (4.5.11) and (4.5.12) in this case.

REMARK 4.5.2 [15]: Applying penalty arguments, the variational inequality (4.5.8) can be replaced by

$$(u,\alpha,\theta) \in K : a_\lambda (u,v-u) \geq P(\beta-\alpha) + M(\delta-\theta) \qquad (4.5.14)$$

$$\text{for every } (v,\beta,\delta) \in K$$

where

$$K = \left\{(v,\beta,\delta) \in V_D \times R \times R : v_N \leq (\beta+s)N_2 + (x_1N_2 - \psi N_1)\delta\right\},$$

$$a_\lambda (u,v) = a(u,v) + \lambda \int_\Omega \operatorname{div} v \, dx. \qquad (4.5.15)$$

REMARK 4.5.3 [14,15]: Introducing Lagrange multiplier $g \in H^{-1/2}(\Gamma)$ such that

$$b(g,(u,\alpha,\theta)) = 0, \quad g \leq 0 \qquad (4.5.16)$$

the variational inequality (4.5.14) can be reduced to saddle point problem

$$(u,\alpha,\theta,g) \in V_D \times R \times R \times Q : \quad a_\lambda(u,v) + \langle g, v_N \rangle = 0,$$

$$b(\bar{g}-g, (u,\alpha,\theta)) \geq 0, \quad P = \langle g, N_2 \rangle$$

$$\text{and } M = \langle g, x_1N_2 - \psi N_1 \rangle$$

$$\text{for every } (v,\beta,\delta,\bar{g}) \in V_D \times R \times R \times Q \qquad (4.5.17)$$

where $\langle .,. \rangle$ is the duality pairing of $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$

and,

$$b(g,(v,\beta,\delta)) = \langle g, v_N - (\beta+s)N_2 - (x_1 N_2 - \psi N_1)\delta \rangle$$

$$Q = \left\{ g \in H^{-1/2}(\Gamma) : g \leq 0 \right\}. \tag{4.5.18}$$

4.6 FINITE ELEMENT DISCRETIZATION:

In the present section the finite element approximation of the variational inequality (4.5.8) is considered. For simplicity, the domain Ω is assumed to be polygonal so that Ω is exactly covered by finite elements.

Let $\{\phi_\alpha\}$ be shape functions to a particular finite element Ω_e, and let the displacement field v be interpolated by v_h defined by

$$v_i^h = v_i^\alpha \phi_\alpha , \quad v_h = \left\{ v_i^h \right\}, \qquad i = 1,2 \cdot \tag{4.6.1}$$

Let ϕ_α be polynomials which include the complete kth order polynomials. Then

$$a(u_h,v_h) = \sum_e a_e(u_h,v_h)$$

$$a_e(u_h,v_h) = v_i^\alpha \overset{(e)}{K}{}_{\alpha\beta}^{ij} u_j^\beta \tag{4.6.2}$$

$$\overset{(e)}{K}{}_{\alpha\beta}^{ij} = \int_{\Omega_e} \mu \left(\delta^{ik} \delta^{jh} + \delta^{ih} \delta^{ki} \right) \phi_{\beta,k} \phi_{\alpha,h} \, dx \cdot$$

In global sense, thus,

$$a(u_h,v_h) = v_i^\alpha \; K_{\alpha\beta}^{ij} \; u_j^\beta \qquad K_{\alpha\beta}^{ij} = \sum_e \overset{(e)}{K}{}_{\alpha\beta}^{ij} \tag{4.6.3}$$

is obtained. Under the above notations, the variational inequality (4.5.8) becomes

$$(u,\alpha,\theta) \in C_h : \quad a(u_h,v_h-u_h) \geq P(\beta-\alpha) + M\,(\delta-\theta)$$
$$\text{for every } (v_h,\beta,\delta) \in C_h \tag{4.6.4}$$

where

$$C_h = \Big\{(v_h,\beta,\delta) \in D_h \times R \times R : v_n^h \leq (\beta+s)n_2+(x_1n_2-\gamma n_1)\delta \text{ on } \Gamma_C\Big\}.$$

$$D_h = \Big\{v_h \in C(\bar{\Omega}) \cap P_k(\Omega) : v_h = 0 \text{ on } \Gamma_D \text{ and } \operatorname{div} v_h = 0 \text{ in } \Omega\Big\}$$

and $P_k(\Omega)$ is the space of all polynomials spanned by shape functions which includes the complete kth order polynomials.

CHAPTER-V

A UNILATERAL CONTACT PROBLEM IN LINEAR ELASTICITY

5.1 INTRODUCTION:

The present chapter is mainly based on the joint work of Brokate and Siddiqi [05]. More precisely this chapter deals with the study of equilibrium of an elastic body when its upper portion is in contact with a nonelastic body of a given weight. The problem is to find the displacements in heavy, linearly elastic body resting on a rigid frictionless horizontal plane. Various aspects of the problem have been discussed by Fichera [10], Villagio [40], Mosco [23], Migno and Puel [22], Duvant and Lions [08], Kikuchi and Oden [14], Nečas and Halaváček [25] and Haslinger and Neitranmäk [12].

The problem is described in Section 5.2 while Section 5.3 is devoted to its variational formations. In Section 5.4 an existence theorem is proved by the theory of Fichera.

5.2 SETTING OF THE PROBLEM:

Let Ω_1, bounded region with smooth boundary $\partial\Omega_1$, in R^3 represents an elastic body and let Ω_2, a bounded region with smooth boundary $\partial\Omega_2$, in R^3 represents a rigid frictionless body placed on the top of Ω_1, Figure 5.2.1. Ω_2 exerts a force P in the vertical downward direction.

The equilibrium position is represented in Figure 5.2.2.

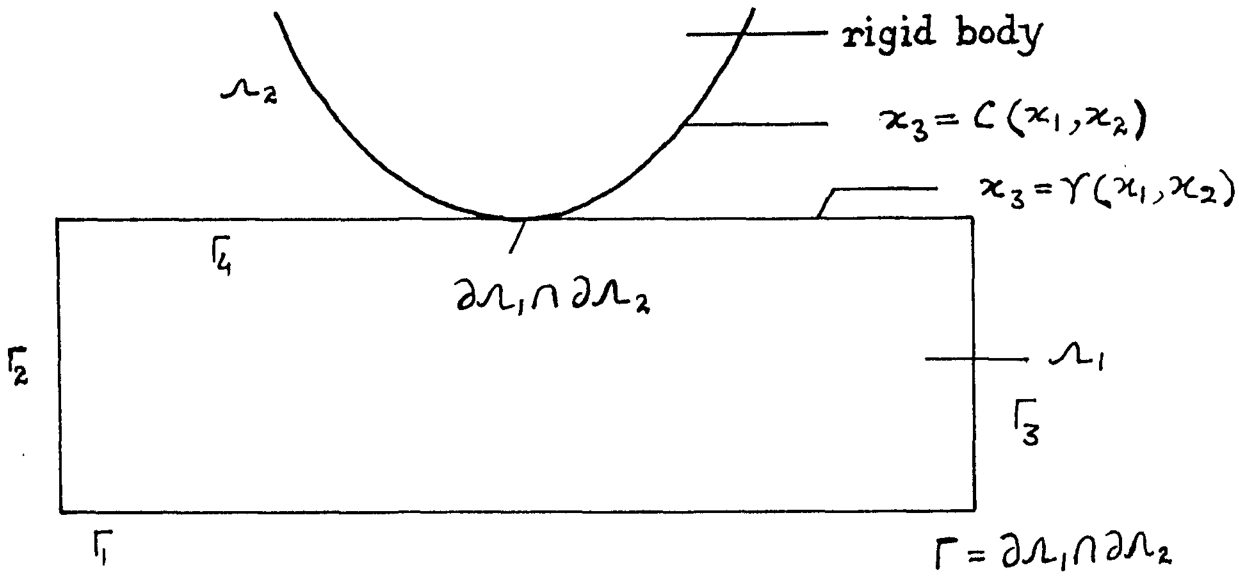

FIG. 5.2.1: Reference configuration.

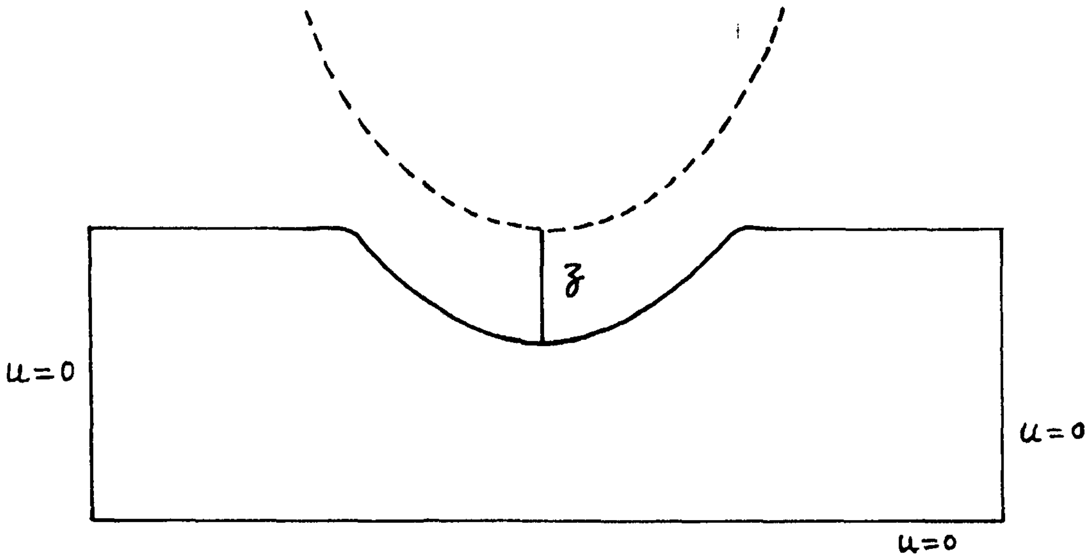

FIG. 5.2.2: Deformed configuration.

$$\partial \Omega_1 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 .$$

Equation of $\partial \Omega_2$ is $x_3 = C(x_1, x_2)$ in the three dimensional coordinate system $(x = (x_1, x_2, x_3))$. Equation of Γ_4 is, $x_3 = \gamma(x_1, x_2)$.

Let $u(x)$, $x = (x_1, x_2, x_3)$, be the displacement in the equilibrium position $(u = (u_1, u_2, u_3) \in R^3)$. If $f = (f_1, f_2, f_3)$ be the body force then by Hooke's law of elasticity in equilibrium position the following system of equations are satisfied:

$$- \sum_{j=1}^{3} \frac{\partial}{\partial x_j} (\sigma_{ij}(u)) = f_i, \ 1 \leq i \leq 3, \quad \text{in } \Omega_1 \qquad (5.2.1)$$

where

$$\sigma_{ij}(u) = \lambda (\sum_{k=1}^{3} \epsilon_{kk}(u) \ \delta_{ij} + 2 \mu \ \epsilon_{ij}(u),$$

$$\epsilon_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

The following boundary conditions are satisfied for the given physical situation,

$$u = 0 \quad \text{on} \quad \partial\Omega_1 \setminus \partial\Omega_1 \cap \partial \Omega_2 \qquad (5.2.2)$$

$$\sigma_{ij}(u) n_j = 0 \quad \text{for all } i \text{ on } \Gamma_4 \setminus \partial\Omega_1 \cap \partial\Omega_2 \qquad (5.2.3)$$

$$i \leq i,j \leq 3$$

where n_j denote the jth component of the unit normal vector to $\partial\Omega_1$.

Let $n = (n_1, n_2, n_3)$ and $t = (0, -n_3, n_2)$ be the unit normal and tangential vectors to $\partial\Omega_1$ respectively.
If $T_n(u) = \sigma_{ij}(u)n_i n_j = \sigma_{ji}(u)n_j n_i = \sigma_{ij}(u)n_j n_i$
and $T_t(u) = \sigma_{ij}(u)n_j t_i$

are normal and tangential components of the stress vector,

$$T(u) = (T_1(u), T_2(u), T_3(u))$$

$$= (\sigma_{1j}(u),\ \sigma_{2j}(u),\ \sigma_{3j}(u)).$$

Then

$$T_t(u) = \sigma_{ij}(u)n_j t_i = 0 \quad \text{on} \quad \partial\Omega_1,\ i \leq i,j \leq 3 \tag{5.2.4}$$

$$T_n(u) = \sigma_{ij}(u)n_i n_j \leq 0 \quad \text{on } \Gamma \tag{5.2.5}$$

$1 \leq i,\ j \leq 3$

$$\sum_{i,j=1}^{3} \int_\Gamma \sigma_{ij}(u)n_j n_i d\Gamma = 0 \qquad \text{on } \Gamma \tag{5.2.6}$$

$$u(x)n(x) + z \leq C(x_1,x_2) - \gamma(x_1,x_2) \text{ on } \Gamma \tag{5.2.7}$$

$$T_n(u)[u(x)n(x) - \{C(x_1,x_2) - \gamma(x_1,x_2) - z\}] = 0. \tag{5.2.8}$$

Finding the solution of the given problem is equivalent to solving (5.2.1) under the conditions (5.2.2) - (5.2.8), that is,

<u>PROBLEM</u> 5.2.1: Find u satisfying (5.2.1) and the conditions (5.2.2.) - (5.2.8).

5.3 <u>VARIATIONAL EQUIVALENCE OF THE PROBLEM</u>:

In the present section we find out a new problem and then proof the equivalence of the new problem with the Problem 5.2.1.

Let

$$V = H^1(\Omega) \times R$$

$$K = \{\hat{u} = (u,z) \in V : u(x)n(x)+z \leq C(x_1,x_2)-\gamma(x_1,x_2)\}$$

$$a(u,v) = \sum_{i,j} \int_{\Omega_1} \sigma_{ij}(u) \, \epsilon_{ij}(v)dx$$

$$= \int_{\Omega_1} [\lambda \operatorname{div}(u)\operatorname{div}(v)+2\mu \sum_{i,j} \epsilon_{ij}(u) \, \epsilon_{ij}(v)]dx \tag{5.3.1}$$

where $\hat{u} = (u,z)$, $\hat{v} = (v,z)$

and $$F(v) = \int_{\Omega_1} f.v \, dx + Pz \tag{5.3.2}$$

PROBLEM 5.3.1: Find $\hat{u} \in K$ such that

$$a(\hat{u}, \hat{v}-\hat{u}) \geq F(\hat{v}-\hat{u}) \qquad \forall\ \hat{v} \in K.$$

THEOREM 5.3.1: Problem 5.2.1 and Problem 5.3.1 are equivalent.

PROOF OF THE THEOREM 5.3.1:

Step 1: Let u be a solution of Problem 5.2.1, then we have

$$-\sum_{j=1}^{3} \frac{\partial}{\partial x_j} (\sigma_{ij}(u)) = f_i.$$

Multiplying both sides by $w_i \in K$ and integrating over Ω_1 we get

$$-\sum_{j=1}^{3} \int_{\Omega_1} \frac{\partial}{\partial x_j} (\sigma_{ij}(u)) w_i dx = \int_{\Omega_1} f_i w_i dx \qquad (5.3.3)$$

L.H.S. of (5.3.3) $= -\sum_{j=1}^{3} \int_{\Omega_1} \frac{\partial}{\partial x_j} (\sigma_{ij}(u)) w_i dx$

$$= \sum_{j=1}^{3} \int_{\Omega_1} \sigma_{ij}(u) \frac{\partial}{\partial x_j} (w_i) dx - \sum_{j=1}^{3} \int_{\partial\Omega_1} \sigma_{ij}(u) w_i n_j d\Gamma$$

$$= I_1 - I_2 \qquad (5.3.4)$$

by Green's Theorem.

Now $I_1 = \sum_{j=1}^{3} \int_{\Omega_1} \sigma_{ij}(u) \frac{\partial}{\partial x_j} (w_i) dx$

$$= \sum_{j=1}^{3} \int_{\Omega_1} \sigma_{ij}(u) \cdot \frac{1}{2} \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right) dx \quad (\text{since } \sigma_{ij}(u) = \sigma_{ij}(u \quad i \leq i, j \leq 3)$$

$$= \sum_{j=1}^{3} \int_{\Omega_1} \sigma_{ij}(u)\ \epsilon_{ij}(w)dx$$

$$= \int_{\Omega_1} [\ \lambda \operatorname{div}(u)\operatorname{div}(u)\ +\ 2\ \mu\ \epsilon_{ij}(u)\ \epsilon_{ij}(w)]dx.$$

In view of (5.2.2) and (5.2.3),

$$I_2 = \sum_{j=1}^{3} \int_{\Gamma} \sigma_{ij}(u) w_i n_j d\Gamma\ .$$

Putting these values of I_1 and I_2 in (5.3.3) and considering summation over i and keeping in mind (5.3.1) and (5.3.2) we get

$$a(\hat{u},\hat{w})\ -\ F(\hat{w})\ =\ \sum_{i,j=0}^{3} \int_{\Gamma} \sigma_{ij}(u) w_i n_j d\Gamma\ .$$

For $\hat{w} = \hat{v}-\hat{u}$, we have

$$a(\hat{u},\hat{v}-\hat{u})\ -\ F(\hat{v}-\hat{u})\ =\ \sum_{i,j=1}^{3} \int_{\Gamma} \sigma_{ij}(u) w_i n_j d\Gamma$$

$$= I_3$$

$\hat{u}$ is a solution of Problem 5.3.1 if we show that $I_3 \geq 0$.

By (5.2.6),

$$I_3 = \sum_{i,j=1}^{3} \int_{\Gamma} \sigma_{ij}(u)(v_i-u_i)n_j d\Gamma + \left\{ \sum_{i,j=1}^{3} \int_{\Gamma} \sigma_{ij}(u) n_j n_i d\Gamma \right\}(z-z_*)$$

We know that $\sum_{j=1}^{3} \sigma_{ij}(u)n_j = \lambda(u)n_i$ where $\lambda(u) \leq 0$ and so

$$I_3 = \int_\Gamma \lambda(u) \sum_{i=1}^{3} (v_i - u_i) n_i d\Gamma + \left\{ \sum_{i=1}^{3} \int_\Gamma \lambda(u) n_i^2 d\Gamma \right\} (z - z_*)$$

$$= \int_\Gamma \lambda(u) \left\{ (v-u)n + (z-z_*) \right\} d\Gamma .$$

We have $v(x)n(x) + z \leq C(x_1,x_2) - \gamma(x_1,x_2)$

$$u(x)n(x) + z_* \leq C(x_1,x_2) - \gamma(x_1,x_2)$$

as $\hat{u} = (u,z)$, $\hat{v} = (v,z) \in K$,

$$(v(x) - u(x))n(x) + (z-z_*) \leq 0 .$$

In view of this result and $\lambda(u) \leq 0$ we see that $I_3 \geq 0$.

<u>Step 2</u>: Let $\hat{u}$ be a solution of Problem 5.3.1, i.e.,

$\hat{u} \in K$

$$a(\hat{u}, \hat{v}-\hat{u}) \geq F(\hat{v}-\hat{u}) \quad \forall \hat{v} \in K .$$

Let $\hat{v} = \hat{u} + \epsilon \hat{w}$, $|\epsilon| < \epsilon_1$, $w_i \in C_0^\infty(\Omega_1)$,

$$a(\hat{u}, \epsilon \hat{w}) \geq F(\epsilon \hat{w})$$

$$a(\hat{u}, \hat{w}) \geq F(\hat{w}) .$$

For $\hat{v} = \hat{u} - \epsilon \hat{w}$, we get

$$a(\hat{u}, -\epsilon \hat{w}) \geq F(-\epsilon \hat{w})$$

$$a(\hat{u}, \hat{w}) \leq F(\hat{w}) .$$

Thus we wee that

$$a(\hat{u},\hat{w}) = F(\hat{w}).$$

This implies

$$\int_{\Omega_1} \sigma_{ij}(u)\ \epsilon_{ij}(w)dx = \int_{\Omega_1} f.wdx + z \sum_{i,j} \int_{\Gamma} \sigma_{ij}(u)n_i n_j d\Gamma$$

$$= \int_{\Omega_1} f.wdx \ (\because z = 0)$$

or

$$\sum_{i,j=1}^{3} \int_{\Omega_1} \sigma_{ij}(u) \frac{\partial w_i}{\partial x_j} dx = \sum_{i=1}^{3} \int_{\Omega_1} f_i\ w_i\ dx$$

or

$$-\sum_{i,j=1}^{3} \int_{\Omega_1} \frac{\partial}{\partial x_j} (\sigma_{ij}(u))w_i dx + \sum_{i,j=1}^{3} \int_{\Gamma} \sigma_{ij}(u)w_i n_j d\Gamma$$

$$= \sum_{i=1}^{3} \int_{\Omega_1} f_i w_i\ dx$$

or $$-\sum_{i,j=1}^{3} \int_{\Omega_1} \frac{\partial}{\partial x_j} (\sigma_{ij}(u))w_i dx = \sum_{i=1}^{3} \int_{\Omega_1} f_i w_i dx \text{ (by (5.2.3))}$$

$$\sum_{j=1}^{3} \int_{\Omega_1} \frac{\partial}{\partial x_j} (\sigma_{ij}(u)) = f_i\ ,$$

that is, u satisfies (5.2.1). (5.2.2) and (5.2.7) are satisfied from the physical situation. To prove (5.2.3), let $z = 0$. Since

$$a(\hat{u},\hat{v}-\hat{u}) \geq F(\hat{v}-\hat{u}) \quad \forall \hat{v} \in K$$

and (5.2.1) is satisfied, we find that

$$\int_{\Gamma} \sigma_{ij}(u) n_j (v_i - u_i) d\Gamma \geq 0.$$

In this relation choose

$$v_i = u_i + w_i \quad \text{and}$$

$$v_i = u_i - w_i, \quad w_i \in C_0^{\infty}(\Omega).$$

Then we get

$$\int_{\Gamma} \sigma_{ij}(u) n_j w_i d\Gamma \geq 0$$

$$-\int_{\Gamma} \sigma_{ij}(u) n_j w_i d\Gamma \geq 0$$

These relations gives as

$$\int_{\Gamma} \sigma_{ij}(u) n_j w_i d\Gamma = 0, \quad \text{or}$$

$$\sigma_{ij}(u) n_j = 0, \quad \forall i .$$

To prove $\sigma_{ij}(u) n_j t_i = 0$ where $t = (0, -n_3, n_2)$.

Since $\sigma_{ij}(u)n_j = 0 \quad \forall\, i,$

$$\sigma_{ij}(u)n_j t_i = 0 \ .$$

To prove $T_n(u) \leq 0$. We know that

$$T_n(u) = \sigma_{ij}(u)n_j n_i \ .$$

Since $\hat{u}$ is a solution of Problem 5.3.1 and (5.2.1) is satisfied, then

$$\sum_{i,j=1}^{3} \int_{\Gamma} \sigma_{ij}(u)\ (v_i-u_i)n_j d\Gamma \geq 0 \qquad \forall\ v_i \in K$$

or
$$\sum_{i,j=1}^{3} \int_{\Gamma} \sigma_{ij}(u)v_i n_j d\Gamma \geq 0.$$

Choose $v_i = \in \xi_i n_i \ , \qquad \xi_i < 0,$ then

$$\sigma_{ij}(u) \in \xi_i n_i n_j \geq 0$$

$$\sigma_{ij}(u)n_i n_j \leq 0$$

$$T_n(u) \leq 0.$$

To prove $T_n(u)[u(x)n(x) - (C(x_1,x_2)-\gamma(x_1,x_2)-z)] = 0.$

If there is no contact then $T_n(u) \leq 0$ and the relation is true. If there is contact, $T_n(u) < 0$ and then

$$u(x)n(x) - (C(x_1,x_2) - \gamma(x_1,x_2)-z) = 0$$

which gives the desired result. This proves the Theorem 5.3.1.

5.4 EXISTENCE OF SOLUTIONS OF THE PROBLEM:

In this section we discuss another problem equivalent to Problem 5.3.1 and then give the existence of solutions of the new problem.

Let $F(v) = \frac{1}{2} a(v,v) - f(v)$ be the functional defined from V into R. Then we have the following minimization problem:

PROBLEM 5.4.1: Find $\hat{u} \in K$ such that

$$F(\hat{u}) \leq F(\hat{v}) \quad \forall \hat{v} \in K$$

where K and V are defined in Section 5.3.

THEOREM 5.4.1: Problems 5.3.1 and 5.4.1 are equivalent.

PROOF OF THEOREM 5.4.1: Suppose

$V = H^1(\Omega) \times R$ is a Hilbert space and

$$K = \{\hat{u} = (u,z) \in H^1(\Lambda) \times R : u(x)n(x)+z \leq C(x_1,x_2)-\gamma(x_1,x_2)\}$$

is closed convex subset of V. Let

$$N(T) = \{\hat{v} \in V: \quad a(\hat{u},\hat{v}) = 0\}$$

$$= \{(0,z): z \in R\}$$

$$= R$$

is a finite dimensional subspace of V and $\theta : V \longrightarrow R$ is a functional such that

$$P = I-\theta$$

where $P : V \longrightarrow H^1(\Lambda)$ is defined by

$$P(v) = v.$$

Then the relation

$$a(\hat{u},\hat{v}) \geq C\|P\hat{v}\| = C\|v\|^2$$

is satisfied and by (1.1) (Fichera [10, p.392] Problem 5.3.1 and Problem 5.4.1 are equivalent.

<u>THEOREM</u> 5.4.2 [05,10]: Problem 5.4.1 has at least one solution.

BIBLIOGRAPHY

[01] ADAMS, R.A., Sobolev Spaces, Academic Press, New York, 1975.

[02] BAIOCCHI, C. and CAPELO, A., Variational and Quasi-variational Inequalities, John Wiley, New York, 1984.

[03] BAIOCCHI, C., BUTTAZZO, G., GASTALDI, F. and TOMARELLI, F., General Existence Theorems for Unilateral Problems in Continuum Mechanics, Arch. Rati. Mech. Anal., 100(1988), 149-189.

[04] BREZZI, F., HAGER, W.W. and RAVIART, P.A., Error Estimates for the Finite Element Solution of Variational Inequalities, Part I: Primary Theory, Numer. Math., 28 (1977), 431-433.

[05] BROKATE, M. and SIDDIQI, A.H., A unilateral Contact Problem and Optimal Control, preprint.

[06] CAMPOS, L.T., ODEN, J.T. and KIKUCHI, N., A Numerical Analysis of a Class of Contact Problems with Friction in Elastostatics, FENOMECH' 81, Part II (Stuttgart, 1981), Comput. Mech. Appl. Mech. Engrg., 34 (1982), 821-845.

[07] CIARLET PHILLIPE, G., The Finite Element Methods for Elliptic Problems, North Holland, Amsterdam, 1978.

[08] DUVANT, G. and LIONS, J.L., Inequalities in Mechanics and Physics, Springer-Verlag, Berlin Heidelberg, New York, 1976.

[09] DUVANT, G. and LIONS, J.L., Elasticite Avec Frottement, J. de Mecanique, 10 (1971), 409-420.

[10] FICHERA, G., Boundary Value Problems in Elasticity with Unilateral Constraints, Encylopedia of Physics ed. S. Flugge, VIa/2, 390-424, Springer-Verlag, Berlin (1972).

[11] GLOWINSKI, R., LIONS, J.L. and TREMOLIERES, R., Numerical Analysis of Variational Inequalities, Studies in Mathematics and its Applications, North Holland Publishing Co. Amesterdam, 1981.

[12] HASLINGER, J. and NEITTANMAK, P., Numerical Functional Analysis and Optimization, 7 (253) (1989/85), 107-124.

[13] HASLINGER, J. and HLAVACEK, Approximation of the Signorini Problem with Friction by a Mixed Finite Element Method, J. Math. Anal. Appl., 86 (1982), 99-122.

[14] KIKUCHI, N. and ODEN, J.T., Contact Problems in Elasticity, SIAM, Phildelphia, U.S.A., 1988.

[15] KIKUCHI, N. and SONG, Y.T., Contact Problems Involving Forces and Moments for Incompressible Linearly Elastic Materials, Internt. J.Sc., 18 (1980), 357-377.

[16] KINDERLEHERER, D. and STAMPACCHIA, G., An Introduction to Variational Inequalities and their Applications, Academic Press, New York, 1980.

[17] KINDERLEHRER, D., Remarks About Signorini's Problems in Linear Elasticity, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (IV) (1981), 605-645.

[18] KINDERLEHRER, D., Estimates for the Solution and its Stability in Signorini's Problems, Appl. Math. Optim. 8 (1982), 159-188.

[19] KREYSZIG, E., Introductory Functional Analysis With Applications, John Wiley and Sons, New York, 1978.

[20] ——————., Advanced Engineering Mathematics, Wiley Eastern Limited, 1983.

[21] LIONS, J.L. and MAGENES, E., Non-Homogeneous Boundary Value Problems and Applications, Springer-Verlag, Berlin, New York, 1972.

[22] MIGNO, F. and PUEL, J.P., Optimal Control in Some Variational Inequalities, SIAM J. Control Optim., 22 (1984), 466-476.

[23] MOSCO, U., Some Introductory Remarks on Implicit Variational Problems, Proc. Regional Workshop on Applicable Mathematics, AMU, 6-16, March, 1989.

[24] ——————., An Introduction to the Approximate Solution of Variational Inequalities, Institute Mathematico G. Castelnuovo Universita Dagli Studi di Roma, 1977-78.

[25] NEČAS, J. and HLÁVAČEK, I., Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction, Elsevier, Amsterdam, 1981.

[26] NOOR, M.A., Variational Inequalities for a Class of Contact Problems with Friction in Elastostatic, C.R. Math. Rep. Acad. Sci. Canada, 5 (1983), 127-133.

[27] ——————., Variational Inequalities Related with a Signorini Problem, C.R. Math. Rep. Acad. Sci. Canada, 7 (1985), 267-272.

[28] ——————., On Class of Variational Inequalities, J. Math. Anal. Appl., 128 (1987), 137-155.

[29] ——————., General Nonlinear Variational Inequalities, J. Math. Anal. Appl., 128 (1987), 78-84.

[30] ODEN, J.T. and KIKUCHI, N., Finite Element Methods for Constrained Problems in Elasticity, TICOM Report 81-10, The University of Texas at Austin (1981).

[31] ODEN, J.T. and CAMPOS, L., Some New Results on Finite Element Methods for Contact Problems with Friction, in New Concepts in Finite Element Analysis, Hughes, Gaslting and Spilker, eds., AMD, Amer. Soc. Mech. Engs., New York, 44 (1980), 1-8.

[32] PANAGIOTOPOULOS, P.D., Inequality Problems in Mechanics and Applications, Convex and Nonconvex Energy Functions, Birkhauser Verlag, Basel/Baston, 1985.

[33] SCARPNI, F., The Use of Maximal Monotone Operator in the Numerical Analysis of Variational Inequalities and Free Boundary Problems, J. Comput. Math., 4 (1986), 266-278.

[34] SIDDIQI, A.H., Functional Analysis with Applications, Tata McGraw Hill Publishing Company, New Delhi (2nd Edition), 1989.

[35] ——————., Lecture Notes on Multivalued Mappings, Regional Workshop on Applicable Mathematics, A.M.U., 6-16, March, 1989.

[36] ——————., Some Remarks on Variational and Quasi-variational Inequalities Monotone Operators, Preprint.

[37] SIDDIQI, A.H., and ANSARI, Q.H., An Algorithm for a Class of Quasivariational Inequalities, J. Math. Anal. Appl., 145 (1990), 413-418.

[38] TARTAR, L., Nonlinear Analysis, Lecture Notes, Orsay Center, Paris University, 1978.

[39] TING, T.W., Generalized Korn's Inequalities, Tensor (1972), 295-302.

[40] VILLAGGIO, P., A Unilateral Contact Problem in Linear Elasticity, J. of Elasticity, 10 (1980), 113-119.